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BIOMETRIKA

STATISTICAL PRINCIPLES OF ROUTINE WORK IN TESTING CLOVER SEED FOR DODDER.

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I. *Introductory.*

The testing of red clover and other small seeds for dodder is one of the important everyday tasks of a Seed Testing Station. As the result of such tests can only be probable and never certain, it is clear that the routine work of these Stations must have a sound statistical basis. The only other writer on this subject that we know of is Schindler*, but we wish to attack the problem from another standpoint.

The samples of clover seeds which are tested contain usually 100—200 gm., or over 50,000 single seed of clover. If in such a sample there are found several dodder seeds (the number varies from 0 to 36 per kg. according to the different countries) this would mean an unfavourable result of the test, and the seed would be considered unsatisfactory. Therefore in sampling clover we are in the position of making very numerous (over 50,000) and practically independent trials, and of counting those trials in which a definite event occurs. Each of these hypothetical trials consists in selecting at random a single seed to form the sample, the event being that the selected seed is a dodder seed.

It is known that if the number of independent trials, n, is very large and the probability, p, of an event, E, in these trials is constant and very small, then the number of occurrences of this event, k, follows the Poisson Law

$$P_k = e^{-m} \frac{m^k}{k!} \quad \ldots\ldots\ldots\

The purpose of this paper is twofold: (i) to show that we are justified in assuming that the number of dodder seeds in samples of clover does follow the Poisson Law and (ii) to construct rules, based on this assumption, which should be followed in routine work in order to minimise the chance of errors in tests for dodder. Some of these rules may have wider application in other sampling problems where the Poisson distribution holds good.

II. *Testing Goodness of Fit of the Poisson Series.*

When testing the assumption that the distribution of dodder in samples of clover seeds follows the Poisson Law we have to keep in mind the circumstances which may cause variations. In all practical cases we are justified in assuming that the probability of a randomly chosen seed being a dodder seed is very small. The only assumption which may be untrue is that of the independency of our "trials." In fact a sample of about 50,000 grains to be tested for dodder is never taken by selecting at random single seeds. On the contrary the samples which are analysed are usually made up of seeds which were in close proximity to each other in the sack. Therefore if the distribution of dodder in the whole mass is not random it will not follow the Poisson Law in the samples. So the problem before us consists really in testing whether the clover seed in ordinary trading conditions is properly mixed or homogeneous with regard to dodder. This has been tested experimentally for us by Broniewski. By means of his experiment we contrived to obtain a sack of clover seed containing a known number of dodder seeds, where the material was mixed to a degree similar to that obtained in seed trading. 100 kg. of red clover seeds were mixed with 2000 dodder seeds; the latter were dyed in order to make them easily distinguishable. In practice the homogeneity of seeds is increased by different processes of threshing, cleaning, etc., then the seed is transported in sacks, often on bad roads, and here the homogeneity may be diminished through continual shaking, the smaller dodder seeds slipping to the bottom. We have tried to allow for this in our experiments. After they were mixed and put into a sack, the seeds were carried in a four-wheeled waggon on a bad road over a distance of about 6 km. Then beginning from the top of the sack, 500 clover samples of about 100 gm. each were taken and examined for dodder. The result was that 986 dodder seeds were found. The small deviation from 1000, which was the expected number of dodder seeds in the 50 kg. analysed, may be regarded as the result of random sampling. The possibility should also be taken into consideration that in the course of the experiment a few dodder seeds may have remained unnoticed and therefore not included in the 986.

In Table I A the empirical results are compared with the theoretical numbers, given by the Poisson Series, assuming that the mathematical expectation $m = 2$, i.e. the *a priori* known mean content of dodder per sample.

Series is of doubtful validity. The origin of negative binomials has been discussed by K. Pearson: *Biometrika*, Vol. XI. pp. 139—144 and especially by "Student": *Ibid.* Vol. XII. pp. 211—215. These papers should certainly be studied by those who simply apply the P, χ^2 test, without consideration of the binomial of which the Poisson Series is supposed to be the limit. ED.]

TABLE I.

Distribution of Dodder in 500 Samples of Clover Seeds. (Our Experiment.)

A.			B.		
$m=2$			$m=1{\cdot}972$		
k	N_k	$N.P_k$	k	N_k	$N.P_k$
0	56	67·67	0	56	69·59
1	156	135·34	1	156	137·23
2	132	135·34	2	132	135·31
3	92	90·22	3	92	88·94
4	37	45·11	4	37	43·85
5	22	18·04	5	22	17·29
6	4	6·02	6	4	5·68
7	0	1·72 } 8·29	7	0	1·60 } 7·79
8	1	0·43	8	1	0·39
over 8	0	0·12	over 8	0	0·12
$n'=7$, $n=n'-1$, $\chi^2=8{\cdot}9169$ $P_{\chi^2}=0{\cdot}179$			$n'=7$, $n=n'-1$*, $\chi^2=8{\cdot}7599$ $P_{\chi^2}=0{\cdot}189$		

k = number of dodder seeds in a sample.
N_k = observed frequency.
$N.P_k$ = expected frequency.
n' = number of groups.
n = degrees of freedom.

In Table I B we have accepted m as unknown, i.e. we have considered these samples as drawn from material, the dodder content of which is entirely unknown to us; thus we estimated m from

$$m=\frac{1}{N}\Sigma kN_k \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

where k is the number of dodder seeds in a 100 gm. sample; N_k the number of samples in which we have found k seeds; N the general number of the investigated samples. Here the approximate value of $m=986/500=1{\cdot}972$.

The diagram corresponds to the numbers given in Table I B.

It will be seen from both tables that the Poisson Law fits extremely well to our data. Thus we may conclude that if the ordinary trading conditions are similar to the conditions of our experiment, which we believe to be true, then the assumption that the frequency of dodder in samples of seeds follows the Poisson Law is generally justified.

* According to the view of R. A. Fisher, E. S. Pearson and J. Neyman the degrees of freedom in this case should be $n'-2$, and accordingly $P_{\chi^2}=0{\cdot}120$, i.e. *less* than P_{χ^2} for $m=2$, which is 0·179.

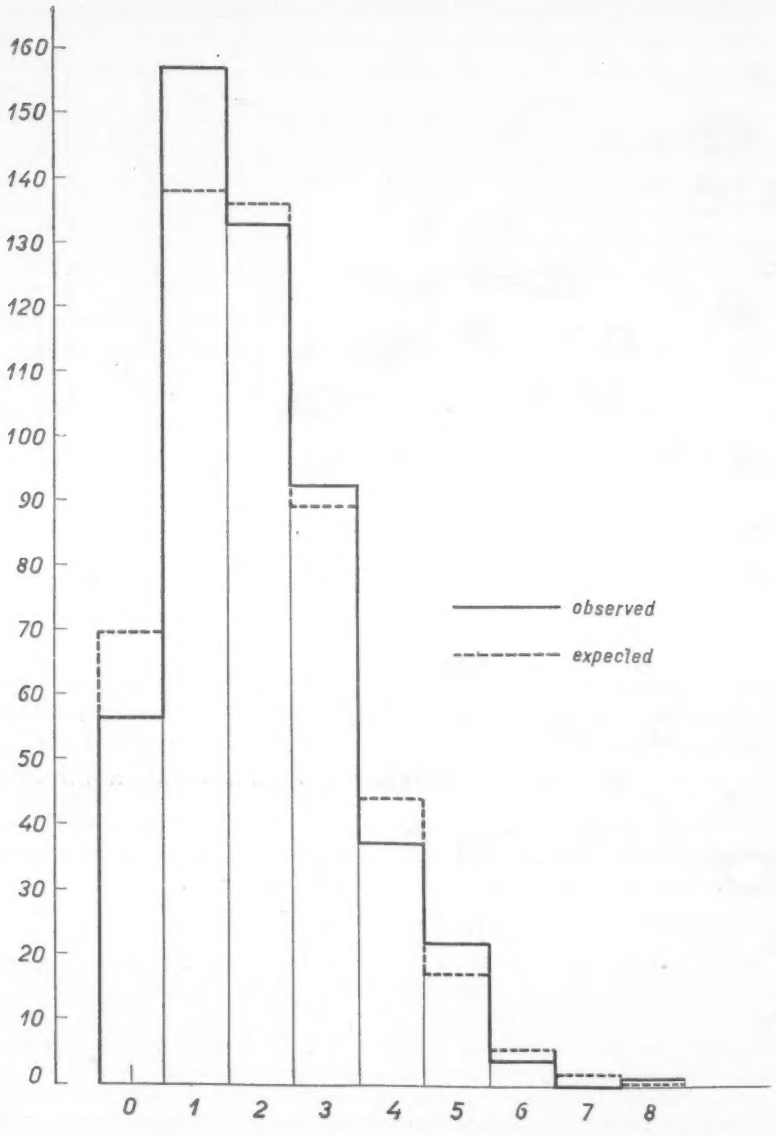

160
150
140
130
120
110
100
90
80
70
60
50
40
30
20
10
0
0 1 2 3 4 5 6 7 8
observed
expected

TABLE II.

Distribution of Dodder in Samples of Clover Seed. (Schindler's Experiments.)

1

k	N_k	$N.P_k$	
0	168	183·94	
1	205	183·94	
2	94	91·97	
3	26	30·66	
4	6	7·66	9·49
5	1	1·53	
over 5	0	0·30	
m		1	
n'		5	
χ^2		5·1990	
P_{χ^2}		0·270	

2

k	N_k	$N.P_k$	
0	599	606·53	
1	315	303·27	
2	74	75·82	
3	12	12·64	14·38
over 3	0	1·74	
m		0·5	
n'		4	
χ^2		0·9848	
P_{χ^2}		0·804	

3

k	N_k	$N.P_k$	
0	382	389·40	
1	111	97·35	
2	7	12·17	13·25
over 2	0	1·08	
m		0·25	
n'		3	
χ^2		5·0027	
P_{χ^2}		0·082	

4

k	N_k	$N.P_k$	
0	284	303·27	
1	170	151·63	
2	39	37·91	
3	7	6·32	7·19
over 3	0	0·87	
m		0·5	
n'		4	
χ^2		3·4863	
P_{χ^2}		0·328	

5

k	N_k	$N.P_k$	
0	795	774·64	
1	94	116·20	
2	11	8·71	9·20
over 2	0	0·45	
m		0·15	
n'		3	
χ^2		5·1286	
P_{χ^2}		0·078	

6

k	N_k	$N.P_k$	
0	447	452·42	
1	51	45·24	
2	2	2·26	2·34
over 2	0	0·08	
m		0·1	
n'		3	
χ^2		0·8477	
P_{χ^2}		0·666	

7

k	N_k	$N.P_k$	
0	473	475·61	
1	26	23·78	
over 1	1	0·61	
m		0·05	
n'		3	
χ^2		0·4709	
P_{χ^2}		0·815	

8

k	N_k	$N.P_k$	
0	295	303·27	
1	153	151·63	
2	44	37·91	
3	8	6·32	7·19
over 3	0	0·87	
m		0·5	
n'		4	
χ^2		1·3075	
P_{χ^2}		0·731	

9

k	N_k	$N.P_k$	
0	22	16·42	
1	29	41·04	
2	55	51·30	
3	43	42·75	
4	34	26·72	
5	10	13·36	
6	3	5·57	8·41
7	4	1·99	
over 7	0	0·85	
m		2·5	
n'		7	
χ^2		8·7617	
P_{χ^2}		0·189	

10

k	N_k	$N.P_k$	
0	0	1·08	
1	3	5·39	
2	13	13·48	
3	15	22·46	
4	33	28·07	
5	28	28·07	
6	24	23·40	
7	21	16·71	
8	10	10·44	
9	8	5·80	10·90
10	0	2·90	
11	4	1·32	
over 11	1	0·88	
m		5	
n'		10	
χ^2		7·0406	
P_{χ^2}		0·633	

11

k	N_k	$N.P_k$	
0	0	0·09	
1	0	0·66	
2	1	2·49	
3	4	6·22	
4	9	11·67	
5	16	17·50	
6	19	21·87	
7	19	23·44	
8	26	21·97	
9	19	18·31	
10	15	13·73	
11	14	9·36	
12	5	5·85	12·68
13	6	3·38	
14	3	1·81	
15	3	0·90	
over 15	1	0·74	
m		7·5	
n'		13	
χ^2		9·8060	
P_{χ^2}		0·633	

k = number of dodder seeds in a sample.
N_k = observed frequency.
$N.P_k$ = expected frequency.

n' = number of groups.
$n'-1$ = degrees of freedom.

Table II gives the results of fitting the Poisson Series to data published by Schindler. The results are often much better than in our case, but this is probably due to the fact that in Schindler's experiments special efforts were made to increase the homogeneity of the material. These efforts increase the agreement between the observed and expected frequencies but diminish the practical importance of the results.

Although we think that the homogeneity of material in our experiment was similar to that in ordinary trading conditions, it is useful to study the dodder distributions in samples drawn by the usual methods from stocks prepared for sale according to the ordinary trading treatment. One set of data of this kind is analysed below. (See p. 281.)

Lastly it should be noted that the independence of samples in Schindler's experiments as well as in ours, was by no means complete. However, the correlation could only be very slight and not have any significant influence on the numerical results. Therefore it will be assumed in the following sections that the number of dodder in samples of clover seed follows exactly the Poisson Law.

III. *Accuracy of the Analysis for Dodder.*

When given seed is analysed for dodder, one of the two following questions may be asked:

(i) Does the dodder content in the seed exceed a certain fixed standard?

(ii) What is the actual dodder content in the seed?

The solution of the first question involves what is called the test of a statistical hypothesis. The other question is an instance of another statistical problem, that of estimation.

(*a*) *Testing the Hypothesis concerning Dodder Content.* After examining a sample of seeds, we want to estimate the average number of dodder seeds in a unit of weight, e.g. in 1 kg. of the stock under consideration. Practically we are only concerned with the question whether that average content does or does not exceed the legal, customary or agreed standard. According to the trading customs accepted in various countries this usual standard varies from 0 to 36 dodder seeds per kg. The accepted standard, let us say, m_0 dodder seeds in 1 kg. of stock, is a limit between two classes of clover seed on the market. This limit will be called the tolerance limit of the true mean dodder content. Stocks containing m_0 or less dodder seeds in a unit of weight belong to one class, while those containing more than m_0 belong to the other class. In each case we are faced with the problem of deciding to which class the material under consideration belongs.

In the theory of testing statistical hypotheses, initiated by J. Neyman and E. S. Pearson*, we distinguish two kinds:

(1) a simple hypothesis which completely specifies the probability law,

(2) a composite hypothesis which specifies the probability law only partially.

* J. Neyman and E. S. Pearson. "On the Problem of the most efficient Tests of Statistical Hypotheses." *Phil. Trans. Roy. Soc.* 1933, Vol. 231 A. pp. 289—337.

It will be seen that when a composite hypothesis is given there are many different ways in which we may complete the specification of the probability law, and thus transform the given composite hypothesis into different simple ones. These hypotheses form a class of simple hypotheses "belonging" to the given composite one.

In our case the statistical hypothesis we test will concern the true mean number of dodder seeds in samples of clover seed. This number will be denoted by m; x will denote the number of dodder seeds in a sample. According to our previous convention it will be assumed that

$$p(x) = e^{-m}\frac{m^x}{x!} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

represents the frequency law of x, given m. Any hypothesis specifying the value of m, say $m = m_i$, will be a simple hypothesis, the probability law $p(x)$ being completely specified. On the other hand if a hypothesis states only that $m > m_i$ or $m \leqslant m_i$ or $m_i < m \leqslant m_j$, then the hypothesis does not specify the probability law completely and is a composite one. Simple hypotheses belonging to the composite which states that $m < 5$ will be all those which ascribe to m any value less than 5, e.g. $m = 1, 1{\cdot}5, \ldots$, etc.

Denote by m_0 the conventional tolerance limit of the dodder content per 100 gm. of seeds. The practical question whether any given seeds contain more or less dodder than the tolerance limit allows may be reduced to the mathematical problem of testing one or other of the following statistical hypotheses. Either we may test the hypothesis $m \leqslant m_0$ or the hypothesis $m > m_0$.

The first of these hypotheses will be called the favourable hypothesis and denoted by F. The other hypothesis that $m > m_0$ will be called unfavourable and denoted by U. Obviously both F and U are composite hypotheses.

When testing statistical hypotheses one must bear in mind two kinds of possible errors: those of the first kind which consist in rejecting the hypothesis when it is true and those of the second kind which consist in accepting the hypothesis when it is false. Neyman and Pearson have developed a theory which allows us to choose from all possible criteria one which is able to control, in the most efficient way, both kinds of errors*. However, our present problem is so simple that there is no need to apply their theory, as the observations only supply us with one number, x, which is the number of dodder seeds in the sample, and which therefore is the only criterion to use.

If we were testing the hypothesis F, stating that $m \leqslant m_0$, we should reject the hypothesis if x, the number of dodder seeds in the sample, is "too large." On the other hand if x is not "too large" we should accept F. If it were the hypothesis U which we were testing we should proceed the other way round and reject U whenever x is "sufficiently small," say $x \leqslant a$. Here we shall only use the Neyman-

* See J. Przyborowski et H. Wileński: "Sur les erreurs de la première et de la seconde catégorie dans la Vérification des Hypothèses concernant la loi de Poisson." *Comptes rendus des séances de l'Académie des Sciences*, Vol. cc. p. 1460. Paris, 1935.

Pearson theory in order to give an exact meaning to the words in inverted commas, namely, "too large" and "sufficiently small." We shall have to take into consideration the probability of both kinds of errors which would correspond to any number which could be chosen to serve as a conventional limit between the values of x which are "too large" and the values which are not "too large."

In the following we shall discuss the problem, assuming that we are testing the hypothesis U. As it was stated this hypothesis will be rejected, and thus the seed will be considered satisfactory, whenever x is "sufficiently small." Denote by a any number which could be used as a conventional limit between the values of x which are "sufficiently small" and those which are not. This will be called the tolerance limit of dodder content in the sample. If a sample contains a dodder seeds or less, the hypothesis U will be rejected.

Let us estimate the probability, say $P_I(m_0, a)$, of first kind errors corresponding to the fixed values of a and m_0. Denote by $p(m)$ the probability law *a priori* concerning the mean number, m, of dodder seeds per sample. Obviously

$$P_I(m_0, a) = \int_{m_0}^{\infty} P\{(x \leqslant a) \mid m\}\, p(m)\, dm \quad \ldots\ldots\ldots\ldots\ldots\ldots (4),$$

where $P\{(x \leqslant a) \mid m\}$ is the probability corresponding to the fixed value of m of having a dodder seeds, at the most, in a sample. In the above formula $p(m)$ is unknown. Therefore the actual value of $P_I(m_0, a)$ is unknown. However, we can find a value which $P_I(m_0, a)$ never exceeds.

Obviously for any value $m > m_0$ we shall have

$$P\{(x \leqslant a) \mid m\} < P\{(x \leqslant a) \mid m_0\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

Therefore

$$P_I(m_0, a) \leqslant P\{(x \leqslant a) \mid m_0\} \int_{m_0}^{\infty} p(m)\, dm \leqslant P\{(x \leqslant a) \mid m_0\} \quad \ldots\ldots\ldots (6)$$

as the integral

$$\int_{m_0}^{\infty} p(m)\, dm \leqslant 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7),$$

whatever the positive number m_0 and whatever the unknown function $p(m)$. It is thus seen that though we are not able to calculate the actual value of the probability of first kind errors, we are able to determine the upper bound of this probability. This is given by

$$P\{(x \leqslant a) \mid m_0\} = \sum_{x \leqslant a} e^{-m_0} \frac{{m_0}^x}{x!} = \frac{1}{a!} \int_{m_0}^{\infty} x^a e^{-x}\, dx \quad \ldots\ldots\ldots\ldots (8).$$

If in fixing the rules of testing the hypothesis U we choose the values of m_0 and a so that $P\{(x \leqslant a) \mid m_0\} = \epsilon$, where ϵ is any small number, say ·05 or ·01, chosen in advance, then we may be certain that in the long run the hypothesis tested will be wrongly rejected with a relative frequency not exceeding ϵ.

Now let us turn to the errors of the second kind which consist in stating that the seed tested is not satisfactory, whereas in fact the average amount of dodder

it contains does not exceed say $m' < m_0$. Denote by $P_{II}(m', a)$ the probability of such an error. Using all previous notation we shall have

$$P_{II}(m', a) = \int_0^{m'} P\{(x > a) \mid m\}\, p(m)\, dm \quad \ldots\ldots\ldots\ldots\ldots\ldots (9),$$

where $P\{(x > a) \mid m\}$ is the probability, corresponding to the fixed value of m, of having more than a dodder seeds in a sample. It is easily seen that owing to the function $p(m)$ being unknown, we are not able to obtain the actual value of $P_{II}(m', a)$. However, as for $m \leqslant m'$,

$$P\{(x > a) \mid m\} \leqslant P\{(x > a) \mid m'\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (10),$$

we may write again

$$P_{II}(m', a) \leqslant P\{(x > a) \mid m'\} \int_0^{m'} p(m)\, dm \leqslant P\{(x > a) \mid m'\} \quad \ldots\ldots (11),$$

as the integral $\int_0^{m'} p(m)\, dm$ cannot exceed unity.

The formulae (6) and (11) give us the limits which the probabilities of first and second kind errors cannot exceed. However, it will be noticed that in unfavourable cases these limits may be approached by the corresponding probabilities as closely as desired, and as in arranging the rules for routine work we have to think of the most unfavourable conditions, we may deal with the upper limits as if they were the values of the probabilities themselves.

(*b*) *Numerical Illustrations.* The values of $P\{(x \leqslant a) \mid m_0\}$ and $P\{(x > a) \mid m'\}$ are given in Tables III and IV respectively*. The use of these tables may be illustrated by the following examples:

Example I. Suppose the tolerance limit m_0 of dodder per sample is fixed at $m_0 = 6$. In other words we assume that if the true average content of dodder per sample in the seed tested exceeds $m_0 = 6$, then it is agreed to consider such a seed unsatisfactory. Suppose further that the rules of testing are set up as follows: the Buyer is willing to accept the seed only when the random sample contains dodder seeds not exceeding, say, $a = 5$. In other cases the Buyer will refuse the seed. Thus the tolerance limit for the true mean dodder content is $m = 6$ and the tolerance limit of the dodder content in the sample is $a = 5$. We now turn to Table III in order to judge whether the interests of the Buyer are properly protected. We find that the value of $P\{(x \leqslant a) \mid m_0\}$ corresponding to these limits is ·446. This means that if the above rules are followed, then unsatisfactory seed will be accepted by the Buyer, with a relative frequency which, in unfavourable cases, may be as large as 44 per cent. It is obvious therefore that if the tolerance limit of the true dodder content is really $m_0 = 6$, the tolerance limit for the dodder content in the sample, fixed at $a = 5$, does not present any real protection to the interests of the Buyer.

* These were calculated using the *Tables of the Incomplete* Γ-*Function*, edited by Karl Pearson and published by the Office of Biometrika, University College, London. Reissue 1934.

TABLE III.

Upper Limits of the Probability of First Kind Errors: $P\{(x \leqslant a) | m_0\}$.

a / m_0	0	1	2	3	4	5	6	7	8	9	10	a / m_0
0·1	·905	—	—	—	—	—	—	—	—	—	—	0·1
0·5	·607	—	—	—	—	—	—	—	—	—	—	0·5
1	·368	·736	—	—	—	—	—	—	—	—	—	1
2	·135	·406	·677	—	—	—	—	—	—	—	—	2
3	·050	·199	·423	·647	—	—	—	—	—	—	—	3
4	·018	·092	·238	·433	·629	—	—	—	—	—	—	4
5	·007	·040	·125	·265	·440	·616	—	—	—	—	—	5
6	·002	·017	·062	·151	·285	·446	·606	—	—	—	—	6
7	·001	·007	·030	·082	·173	·301	·450	·599	—	—	—	7
8	—	·003	·014	·042	·100	·191	·313	·453	·593	—	—	8
9	—	·001	·006	·021	·055	·116	·207	·324	·456	·587	—	9
10	—	—	·003	·010	·029	·067	·130	·220	·333	·458	·583	10
11	—	—	·001	·005	·015	·038	·079	·143	·232	·341	·460	11
12	—	—	·001	·002	·008	·020	·046	·090	·155	·242	·347	12
13	—	—	—	·001	·004	·011	·026	·054	·100	·166	·252	13
14	—	—	—	—	·002	·006	·014	·032	·062	·109	·176	14
15	—	—	—	—	·001	·003	·008	·018	·037	·070	·118	15
16	—	—	—	—	—	·001	·004	·010	·022	·043	·080	16
17	—	—	—	—	—	·001	·002	·005	·013	·026	·049	17
18	—	—	—	—	—	—	·001	·003	·007	·015	·030	18
19	—	—	—	—	—	—	·001	·002	·004	·009	·018	19
20	—	—	—	—	—	—	—	·001	·002	·005	·011	20
21	—	—	—	—	—	—	—	—	·001	·003	·006	21
22	—	—	—	—	—	—	—	—	·001	·002	·004	22
23	—	—	—	—	—	—	—	—	—	·001	·002	23
24	—	—	—	—	—	—	—	—	—	—	·001	24
25	—	—	—	—	—	—	—	—	—	—	·001	25

a = tolerance limit for the dodder content in the sample.
m_0 = " " " " true mean dodder content.

Looking at Table III it will be seen that keeping the tolerance limit of the true mean dodder content at $m_0 = 6$, we may procure a reasonable level of safety for the Buyer if we fix the tolerance limit of dodder content in the sample at $a = 2$ or even $a = 1$. This means that the seed will be accepted as satisfactory whenever the dodder content in the sample does not exceed 2 or 1 seeds. Even with such precautions the Buyer may get unsatisfactory goods. This, however, will happen but rarely and with a relative frequency not exceeding ·06 in the one case and ·02 in the other.

Example II. We have considered the problem of tolerance limits from the point of view of the Buyer, now let us consider it from the Seller's.

Assume that the tolerance limit for dodder content in the sample has been fixed at $a = 2$ to suit the interests of the Buyer who is not willing to accept seed with a true mean content of dodder exceeding $m_0 = 6$ seeds per size of sample. The Seller will naturally be interested in the chance that the seed, which actually contains less than 6 dodder seeds per sample, should successfully pass the test.

The probabilities of second kind errors, i.e. that the seed will be unjustly rejected, are given in Table IV, where a is again the tolerance limit for the sample and m' the assumed true mean content of dodder in the seed tested.

TABLE IV.

Upper Limits of the Probability of Second Kind Errors: $P\{(x > a)|m'\}$.

m' / a	·05	·1	·2	·25	·5	1	2	2·5	5	10	20	25	m' / a
0	·049	·095	·181	·221	·393	·632	·865	·918	·993	—	—	—	0
1	·001	·005	·018	·027	·090	·264	·594	·713	·960	·999	—	—	1
2	—	—	·001	·002	·014	·080	·323	·456	·875	·997	—	—	2
3	—	—	—	—	·002	·019	·143	·242	·735	·990	—	—	3
4	—	—	—	—	—	·004	·053	·109	·560	·971	—	—	4
5	—	—	—	—	—	·001	·017	·042	·384	·933	—	—	5
6	—	—	—	—	—	—	·005	·014	·238	·870	—	—	6
7	—	—	—	—	—	—	·001	·004	·133	·780	·999	—	7
8	—	—	—	—	—	—	—	·001	·068	·667	·998	—	8
9	—	—	—	—	—	—	—	—	·032	·542	·995	—	9
10	—	—	—	—	—	—	—	—	·014	·417	·989	—	10
11	—	—	—	—	—	—	—	—	·005	·303	·979	·999	11
12	—	—	—	—	—	—	—	—	·002	·208	·961	·997	12
13	—	—	—	—	—	—	—	—	·001	·136	·934	·994	13
14	—	—	—	—	—	—	—	—	—	·084	·895	·988	14
15	—	—	—	—	—	—	—	—	—	·049	·843	·978	15
16	—	—	—	—	—	—	—	—	—	·027	·779	·962	16
17	—	—	—	—	—	—	—	—	—	·014	·703	·940	17
18	—	—	—	—	—	—	—	—	—	·007	·619	·908	18
19	—	—	—	—	—	—	—	—	—	·003	·530	·866	19
20	—	—	—	—	—	—	—	—	—	·002	·441	·815	20
21	—	—	—	—	—	—	—	—	—	·001	·356	·753	21
22	—	—	—	—	—	—	—	—	—	—	·279	·682	22
23	—	—	—	—	—	—	—	—	—	—	·220	·606	23
24	—	—	—	—	—	—	—	—	—	—	·157	·527	24
25	—	—	—	—	—	—	—	—	—	—	·112	·447	25
26	—	—	—	—	—	—	—	—	—	—	·078	·371	26
27	—	—	—	—	—	—	—	—	—	—	·053	·300	27
28	—	—	—	—	—	—	—	—	—	—	·034	·237	28
29	—	—	—	—	—	—	—	—	—	—	·022	·182	29
30	—	—	—	—	—	—	—	—	—	—	·014	·137	30
31	—	—	—	—	—	—	—	—	—	—	·008	·100	31
32	—	—	—	—	—	—	—	—	—	—	·005	·072	32
33	—	—	—	—	—	—	—	—	—	—	·003	·050	33
34	—	—	—	—	—	—	—	—	—	—	·001	·034	34
35	—	—	—	—	—	—	—	—	—	—	·001	·023	35
36	—	—	—	—	—	—	—	—	—	—	—	·015	36
37	—	—	—	—	—	—	—	—	—	—	—	·009	37
38	—	—	—	—	—	—	—	—	—	—	—	·006	38
39	—	—	—	—	—	—	—	—	—	—	—	·003	39
40	—	—	—	—	—	—	—	—	—	—	—	·002	40
41	—	—	—	—	—	—	—	—	—	—	—	·001	41
42	—	—	—	—	—	—	—	—	—	—	—	·001	42

a = tolerance limit for dodder content in a sample.

m' = assumed true mean dodder content per size of sample.

TABLE IV (*continued*).

$m'=50$		$m'=100$				$m'=200$			
a		a		a		a		a	
29	·999	71	·999	103	·358	158	·999	203	·398
30	·998	72	·998	104	·322	159	·998	204	·371
31	·997	73	·997	105	·287	160	·998	205	·345
32	·996	74	·996	106	·255	161	·997	206	·320
33	·993	75	·995	107	·225	162	·997	207	·295
34	·989	76	·993	108	·196	163	·996	208	·271
35	·984	77	·990	109	·171	164	·995	209	·249
36	·976	78	·987	110	·147	165	·994	210	·227
37	·966	79	·983	111	·126	166	·992	211	·207
38	·953	80	·978	112	·107	167	·991	212	·188
39	·935	81	·971	113	·091	168	·989	213	·170
40	·914	82	·964	114	·076	169	·986	214	·153
41	·888	83	·954	115	·063	170	·983	215	·137
42	·856	84	·943	116	·052	171	·980	216	·122
43	·820	85	·930	117	·043	172	·976	217	·109
44	·779	86	·915	118	·035	173	·972	218	·097
45	·733	87	·897	119	·028	174	·966	219	·086
46	·683	88	·877	120	·023	175	·961	220	·075
47	·630	89	·854	121	·018	176	·954	221	·066
48	·575	90	·829	122	·014	177	·946	222	·058
49	·518	91	·802	123	·011	178	·938	223	·050
50	·462	92	·772	124	·009	179	·928	224	·044
51	·408	93	·740	125	·007	180	·918	225	·038
52	·355	94	·705	126	·005	181	·906	226	·032
53	·304	95	·669	127	·004	182	·893	227	·028
54	·258	96	·632	128	·003	183	·879	228	·024
55	·216	97	·593	129	·002	184	·865	229	·020
56	·178	98	·554	130	·002	185	·848	230	·017
57	·145	99	·514	131	·001	186	·830	231	·015
58	·117	100	·474	132	·001	187	·811	232	·012
59	·092	101	·434	133	·001	188	·791	233	·010
60	·072	102	·396			189	·769	234	·009
61	·056					190	·747	235	·007
62	·042					191	·723	236	·006
63	·032					192	·699	237	·005
64	·024					193	·674	238	·004
65	·017					194	·647	239	·003
66	·013					195	·621	240	·003
67	·009					196	·593	241	·002
68	·006					197	·566	242	·002
69	·004					198	·537	243	·001
70	·003					199	·510	244	·001
71	·002					200	·481	245	·001
72	·001					201	·453	246	·001
73	·001					202	·425	247	·001

It will be seen that if the seed tested contains, on the average, as many as $m'=5$ grains per sample, then the probability of the seed not being passed by the test is equal to ·875. In other words, if the interests of the Buyer are met by choosing the sample tolerance limit $a=2$, then in 87·5 per cent. of cases the seed offered will be found unsatisfactory, showing more than two dodder seeds in a

sample, when in fact it contains $m' = 5$ dodder seeds per sample, thus satisfying the requirements of the Buyer. A reasonable chance of passing the test successfully corresponds to $m' = \cdot 5$. Here the probabilities of an unjust rejection of the seed is $\cdot 014$.

Thus in order to secure a reasonable chance of his goods passing the test, the Seller must aim at a dodder content about 12 times smaller than that which is fixed as the tolerance limit $m_0 = 6$.

The position is most unsatisfactory, as whenever the test is efficiently protecting the interest of one of the contracting parties, the interests of the other are seriously damaged.

Now let us assume that the samples tested are, say three times, larger so that the tolerance limit of the true mean dodder content is correspondingly increased to $m_0 = 18$. It will be seen that a reasonable protection of the Buyer's interests will be attained if the tolerance limit of the dodder content in the sample were fixed at, say, $a = 10$. Doing so we may be certain that the corresponding probability of first kind of error will not exceed $\cdot 03$. (See Table III.) On the other hand, looking at Table IV, we see that the seed, whose true mean dodder content $m' = 5$, will have a reasonable chance of passing the test, the upper limit of the probability of second kind error being $\cdot 014$. Now the ratio $m_0/m' = 3 \cdot 6$, and thus the gap between the requirement of the Buyer and the safety level of the Seller is shortened very considerably. It will be seen that the use of larger samples is most beneficial both to the Seller and the Buyer. Unfortunately this means a larger expenditure on analysis. However, we shall make further mention of this point below.

Example III. We shall here call special attention to the case when the tolerance limit of dodder content in the sample is fixed at the lowest possible value $a = 0$. It is interesting to see exactly what sort of protection this provides to the interests of the Buyer. Looking at Table III we find that if $m_0 = 3$, then, on the average, one sample out of 20 would be totally free from dodder. The test applied will thus give a favourable result and the seed will be accepted. We see therefore that statements like "dodder free" if they are based upon analysis of only one sample of usual size may be frequently misleading. The seed which is described as "dodder free" is quite likely to contain as many as two dodder seeds per size of the sample tested, it is also quite possible that it contains three; in fact we should only be surprised when the true mean content is as large or larger than four.

If the size of the sample tested is large, then a dodder content of 2 or 3 grains per sample could be perhaps considered as negligible. But in the ordinary size of sample such amounts would be disappointing, especially in a seed which has been qualified as dodder free.

(c) *Upper Confidence Limits of Dodder Content.* The last example leads us to the new form of problem of estimation which has been started by R. A. Fisher* and

* R. A. Fisher: *Proc. Cam. Phil. Soc.* Vol. XXVI. part 4.

then developed by J. Neyman*, namely the problem of confidence limits for a given unknown parameter. In our case the role of this parameter is played by the true mean dodder content per size of sample in a given consignment of seed.

We think that the best way of expressing the results of an analysis for dodder content consists in stating the limit which presumably is not exceeded by the true mean dodder content, e.g. the seed does not contain more than, say, 4 grains of dodder per kg. The limit stated, viz. 4, is what is called the upper confidence limit. J. Neyman has shown how statements of this kind can be made so that the probability of their being correct will not be smaller than a limit fixed in advance. This limit of probability has been called "the confidence coefficient." If it is said that a seed testing station works at a confidence coefficient, say $\alpha = \cdot 99$, it means that its verdicts concerning the quality of the seed in the form stated above may be wrong only with a relative frequency not exceeding $1 - \alpha$ or one in a hundred.

The upper confidence limits, corresponding to the chosen confidence coefficient α, are found as follows: denote by a any integer number of dodder seeds which may be found in a sample, and find such a value of $m = m_a$ that

$$P\{(x > a) \mid m_a\} = \alpha \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12).$$

The numbers m_a thus defined are the upper confidence limits. Suppose, in fact, that the above equation is solved for any $a = 0, 1, 2, \ldots$, and suppose we make a rule of stating that the true mean dodder content, m, in seed does not exceed m_a whenever the analysis of a sample gives $x = a$. It is easy to see that the probability of an error in such a statement never exceeds the limit $1 - \alpha$.

This may be proved as follows†: we notice first that an error in estimating the dodder content by stating the upper confidence limit may occur only if the actual dodder content m and the dodder content in the sample satisfy certain inequalities. Namely, we shall be wrong whenever, say,

$$m_a < m \leqslant m_{a+1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

and

$$x \leqslant a \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14)$$

happen at the same time.

Let us estimate the probability of the simultaneous occurrence of the above two inequalities. For this purpose we notice that

$$P\{(x \leqslant a) \mid m\} = 1 - P\{(x > a) \mid m\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (15).$$

If $m_a < m$, then obviously

$$P\{(x > a) \mid m\} > P\{(x > a) \mid m_a\} = \alpha \quad \ldots\ldots\ldots\ldots\ldots\ldots (16).$$

Thus we shall have

$$P\{(x \leqslant a) \mid m\} < 1 - \alpha \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (17).$$

This being true for any m satisfying (13) and for any a, we conclude that whatever the unknown frequency distribution of m, the probability of having (13) and (14)

* J. Neyman: *Journal R. Statistical Soc.* Vol. xcvii. pp. 589—593.

† See T. Matuszewski, J. Neyman and J. Supińska: *Supplement to the Journal R. Statistical Soc.* Vol. ii. part 1.

simultaneously does not exceed $1-\alpha$. This is another way of saying that the probability of errors in statements $m \leqslant m_a$ made when we get, in the sample, exactly $x = a$ dodder seeds, does not exceed $1-\alpha$.

The upper confidence limits for the mean dodder content corresponding to different confidence coefficients, α, are given in Table V*. The use of this table may be illustrated in the following example:

Example IV. Suppose in three different samples of 100 gm. of clover seeds the amount of dodder found was $x_1 = 0$, $x_2 = 3$ and $x_3 = 10$ respectively. The statement concerning the true mean dodder content in the seeds tested must depend upon the caution of the testing station. If it be considered that errors in statements occurring, on the average, once in ten times may be admitted, then the confidence coefficient chosen must be ·9. In this case the statements concerning the true dodder content per 100 gm. of seeds in the consignments tested, should be

$$m_1 \leqslant 2{\cdot}3, \quad m_2 \leqslant 6{\cdot}7, \quad m_3 \leqslant 15{\cdot}4.$$

(In order to obtain the estimates per 1 kg. we have only to multiply the above figures by 10.)

However, it may be considered that this frequency of errors, i.e. 1 in 10, is too considerable. Then the confidence coefficient chosen should be larger. It may be desired to commit errors as seldom as one in a hundred, then $\alpha = {\cdot}99$ and our statements about the dodder content in the same consignments would be

$$m_1 \leqslant 4{\cdot}6, \quad m_2 \leqslant 10{\cdot}0, \quad m_3 \leqslant 20{\cdot}2.$$

We notice that the above example suggests again that the reliability of tests for dodder requires the use of larger samples. In fact, even when no dodder seeds are found in a sample, if we do not want to risk an error more than once in a hundred statements, we should only state that the true dodder content in the seed does not exceed 4·6 grains per size of sample. If this is large then the accuracy may be sufficient; if, however, the sample is small then it may be considered unsatisfactory.

The question of the size of the sample may be treated more accurately in the following way:

Fix any confidence coefficient, say $\alpha = {\cdot}99$, and suppose that the samples in the above examples contained 100 gm. of seeds. The result of the test of the first sample may be stated by saying that in the seed analysed there was probably no more than 46 grains of dodder per 1 kg.

Now assume that the sample contained 5 kg. of seed, and that again no dodder was found. Our estimate per size of sample will be the same, i.e. not more than 4·6 grains on the average. But if we turn from the size of sample to the standard unit of 1 kg., the statement will change. We shall state, in fact, that in the consignment there is presumably no more than ·9 grain of dodder per 1 kg. In this way it is easy to calculate that if for the sample which proves dodder free it should be sufficient to state that the amount of dodder per 1 kg. of seed does not exceed 1 grain, then the size of this sample should be as large as 4·6 kg.

* This was again calculated using the *Tables of the Incomplete* Γ*-Function.*

TABLE V.

Upper Confidence Limits, m_a, for the True Mean Dodder Content.

a / a	0·9	0·95	0·98	0·99	0·995	0·999	a / a
0	2·3	3·0	3·9	4·6	5·3	6·9	0
1	3·9	4·7	5·8	6·6	7·4	9·2	1
2	5·3	6·3	7·5	8·4	9·3	11·2	2
3	6·7	7·8	9·1	10·0	11·0	13·1	3
4	8·0	9·2	10·6	11·6	12·6	14·8	4
5	9·3	10·5	12·0	13·1	14·2	16·5	5
6	10·5	11·8	13·4	14·6	15·7	18·1	6
7	11·8	13·1	14·8	16·0	17·1	19·6	7
8	13·0	14·4	16·2	17·4	18·6	21·2	8
9	14·2	15·7	17·5	18·8	20·0	22·7	9
10	15·4	17·0	18·8	20·2	21·4	24·1	10
11	16·6	18·2	20·1	21·5	22·8	25·6	11
12	17·8	19·4	21·4	22·8	24·1	27·0	12
13	19·0	20·7	22·7	24·1	25·5	28·4	13
14	20·1	21·9	24·0	25·5	26·8	29·9	14
15	21·3	23·1	25·2	26·7	28·2	31·2	15
16	22·5	24·3	26·5	28·0	29·5	32·6	16
17	23·6	25·5	27·8	29·3	30·8	34·0	17
18	24·8	26·7	29·0	30·6	32·1	35·4	18
19	25·9	27·9	30·2	31·9	33·4	36·7	19
20	27·0	29·1	31·5	33·1	34·7	38·0	20
21	28·2	30·2	32·7	34·4	36·0	39·4	21
22	29·3	31·4	33·9	35·6	37·2	40·7	22
23	30·5	32·6	35·1	36·8	38·5	42·0	23
24	31·6	33·8	36·3	38·1	39·8	43·3	24
25	32·7	34·9	37·5	39·3	41·0	44·6	25
26	33·8	36·1	38·7	40·5	42·3	45·9	26
27	35·0	37·2	39·9	41·8	43·5	47·2	27
28	36·1	38·4	41·1	43·0	44·7	48·5	28
29	37·2	39·5	42·3	44·2	46·0	49·8	29
30	38·3	40·7	43·5	45·4	47·2	51·1	30
31	39·4	41·8	44·7	46·6	48·4	52·3	31
32	40·6	43·0	45·8	47·8	49·7	53·6	32
33	41·7	44·1	47·0	49·0	50·9	54·9	33
34	42·8	45·3	48·2	50·2	52·1	56·2	34
35	43·9	46·4	49·4	51·4	53·3	57·4	35
36	45·0	47·5	50·5	52·6	54·6	58·7	36
37	46·1	48·7	51·7	53·8	55·8	59·9	37
38	47·2	49·8	52·9	55·0	57·0	61·2	38
39	48·3	50·9	54·0	56·2	58·2	62·4	39
40	49·4	52·1	55·2	57·4	59·4	63·7	40
41	50·5	53·2	56·4	58·5	60·6	64·9	41
42	51·6	54·3	57·5	59·7	61·8	66·1	42
43	52·7	55·5	58·7	60·9	63·0	67·4	43
44	53·8	56·6	59·8	62·1	64·2	68·6	44
45	54·9	57·7	61·0	63·2	65·4	69·8	45
46	56·0	58·8	62·1	64·4	66·5	71·1	46
47	57·1	59·9	63·3	65·6	67·7	72·3	47
48	58·2	61·1	64·4	66·8	68·9	73·5	48
49	59·3	62·2	65·6	67·9	70·1	74·7	49
50	60·4	63·3	66·7	69·1	71·3	76·0	50

a = dodder content in a sample.
α = confidence coefficient.

Calculations of this sort have been performed for six different values of the confidence coefficient for several values of a or the sample dodder content, and for several values of m'_a or the upper confidence limit of dodder content *per* 1 *kg.* These results are presented in Table VI. The use of this table may be illustrated in the following example:

Example V. Suppose that the confidence coefficient has been fixed at $\alpha = \cdot 99$ and we are interested in the size of the sample which, if it contains say $a = 3$ dodder seeds, would entitle us to state that the true dodder content in the whole consignment does not exceed $m_a' = 5$ grains per 1 kg. The size of the sample required will be found in Table VI in the column corresponding to $\alpha = \cdot 99$, and in the row corresponding to $a = 3$ and $m_a' = 5$. This is seen to be 2 kg. If the confidence coefficient had been fixed at $\alpha = \cdot 95$, then the amount of seed forming the sample could be smaller, namely 1·56 kg.

TABLE VI.

Size of Samples to be tested assuring sufficient Accuracy in the Estimate of the True Dodder Content per 1 *kg.*

a	α / m_a'	0·9	0·95	0·98	0·99	0·995	0·999
0	0·1	23·00 kg.	30·00 kg.	40·00 kg.	46·00 kg.	53·00 kg.	69·00 kg.
	1	2·30 „	3·00 „	4·00 „	4·60 „	5·30 „	6·90 „
	5	0·46 „	0·60 „	0·80 „	0·92 „	1·06 „	1·38 „
	10	0·23 „	0·30 „	0·40 „	0·46 „	0·53 „	0·69 „
	20	0·12 „	0·15 „	0·20 „	0·23 „	0·27 „	0·35 „
	40	0·06 „	0·08 „	0·10 „	0·12 „	0·13 „	0·17 „
1	5	0·78 „	0·94 „	1·16 „	1·32 „	1·48 „	1·84 „
	10	0·39 „	0·47 „	0·58 „	0·66 „	0·74 „	0·92 „
	20	0·20 „	0·24 „	0·29 „	0·33 „	0·37 „	0·46 „
	40	0·10 „	0·12 „	0·15 „	0·17 „	0·19 „	0·23 „
2	5	1·06 „	1·26 „	1·50 „	1·68 „	1·86 „	2·24 „
	10	0·53 „	0·63 „	0·75 „	0·84 „	0·93 „	1·12 „
	20	0·27 „	0·32 „	0·38 „	0·42 „	0·46 „	0·56 „
	40	0·13 „	0·16 „	0·19 „	0·21 „	0·23 „	0·28 „
3	5	1·34 „	1·56 „	1·82 „	2·00 „	2·20 „	2·62 „
	10	0·67 „	0·78 „	0·91 „	1·00 „	1·10 „	1·31 „
	20	0·33 „	0·39 „	0·45 „	0·50 „	0·55 „	0·65 „
	40	0·17 „	0·19 „	0·23 „	0·25 „	0·27 „	0·33 „
50	5	12·08 „	12·66 „	13·34 „	13·82 „	14·26 „	15·20 „
	10	6·04 „	6·33 „	6·67 „	6·91 „	7·13 „	7·60 „
	20	3·02 „	3·16 „	3·34 „	3·45 „	3·56 „	3·80 „
	40	1·51 „	1·58 „	1·67 „	1·73 „	1·78 „	1·90 „

a = number of dodder seeds in a sample.
m_a' = upper confidence limit of dodder content per 1 kg.
α = confidence coefficient.

IV. *Practical Conclusions.*

We have seen above that if the tolerance limit for the true mean dodder content is comparatively low we have to face the fact that the method of testing is liable to greatly injure the interests either of the Buyer or of the Seller or both. This position is improved if the tolerance limit for the true dodder content is high. In order to work with high tolerance limits we have either to consider clover seed containing larger amounts of dodder as satisfactory, which is impossible, or to work with larger samples.

It appears that it is extremely difficult to give reliable statements accurately describing the dodder content in seed, if these statements are to apply to small quantities of seeds such as a sack, as the size of the sample taken out of each sack which would provide sufficiently accurate information would have to be so large that the analysis would be too costly. A possible issue seems to consist in estimating the dodder content in larger quantities of seed such as 100 sacks.

If such an amount of seed is presented and there are grounds to believe that it is homogeneous, a seed station may undertake the analysis of the whole. Comparatively small samples may be taken out of each sack and analysed separately. The data obtained must be used for two purposes: (i) to test the hypothesis that the 100 sacks present, in fact, homogeneous seed, and (ii) to estimate the dodder content in the seed. If the result of (i) is favourable and the seed is proved to be homogeneous there will be only one estimate concerning the whole consignment. If however the result of (i) is unfavourable, it will be necessary to estimate the dodder content separately for each sack.

The homogeneity test, mentioned in (i), is carried out in exactly the same way as that which led us to Table I B. If the seed proves homogeneous then the single estimate of the true mean dodder content will be obtained by combining all 100 samples from the separate sacks into one large sample. As the sum of a number of variables each following the Poisson Law of Frequency itself follows the same Law all our tables will be valid to judge the accuracy of this estimate. Practically the work will consist in summing up the amount of dodder found in separate samples and in referring to Table V to get the upper confidence limit for the mean dodder content in the whole consignment per weight equal to the size of the 100 samples combined. Dividing the upper confidence limit thus obtained by 100 we shall obtain the confidence limit of mean dodder content per size of one sample.

Obviously, even if the requirements as to the purity of clover seed are very strict, owing to the large number of partial samples from separate sacks, when dealing with the combined estimate we shall have to work with high tolerance limits. Thus the interests of both Buyer and Seller will be protected as far as possible.

It must be emphasised that the final verdict concerning the dodder content will apply to the whole stock of 100 sacks of seed. Among those 100 sacks there

will be several showing apparently smaller dodder content and others showing larger. However, it would not be right to conclude that these sacks differ in reality in their mean dodder content, as formerly it has been found that, when analysing the distribution of dodder content in samples from single sacks, the variation is no more than what should be expected from purely random causes when dealing with perfectly homogeneous material. In other words, we shall have to assume that if a sample from one sack contained less dodder than that from another, this difference is due solely to random variation and not to a real difference in the dodder content of the two sacks.

Assume now another eventuality: namely, that the χ^2 test showed a marked heterogeneity of seed in the 100 sacks presented for analysis. In this case the Seed Station should withhold from a general verdict concerning the whole stock, as this should be considered as heterogeneous. Separate estimates of the dodder content should be given for single sacks. These, of course, being based on the analysis of small samples, will be generally rather inaccurate, but we do not think there are any means of improving the position. It will perhaps be possible to increase the samples from single sacks, but this involves considerable expense.

Let us illustrate the above procedure by an example. A stock of 100 sacks of clover seed has lately been analysed in the seed testing laboratory in our Department. A sample of 100 gm. of seed was taken out of each sack and analysed for dodder. Columns 1 and 2 of Table VII state the results of the analysis. It will be seen that 76 samples proved to be dodder free, 22 others contained 1 dodder seed, etc. According to the principles stated above we tested the hypothesis that

TABLE VII.

Distribution of Dodder Content in Samples from 100 *Separate Sacks.*

k	N_k	$N.P_k$
0	76	77·11
1	22	20·05
2	2	2·61 } 2·84
over 2	0	0·23 }
$m=0{\cdot}26$, $n'=3$, $\chi^2=0{\cdot}4541$, $P_{\chi^2}=0{\cdot}821$		

k = number of dodder seeds in a sample.
N_k = observed frequency.
$N.P_k$ = expected frequency.
n' = number of groups.
$n'-1$ = degrees of freedom*.

* According to the view of R. A. Fisher, E. S. Pearson and J. Neyman the degrees of freedom in this case should be $n'-2$ and $P_{\chi^2}=\cdot500$.

the stock analysed was homogeneous. Column 3 of Table VII gives the frequencies calculated from the fitted Poisson Series. It will be seen that the fit is almost perfect, $P = \cdot 821$ (or $\cdot 50$, if degrees of freedom $= 1$). Thus we can conclude that there are no difficulties in formulating a verdict concerning the whole stock of seed as the differences between dodder content in samples from single sacks will be ascribed solely to individual random variation.

In all the 100 samples of 100 gm. each 26 dodder seeds have been found; thus the dodder content is 2·6 dodder seeds per kg. Choosing the confidence coefficient ·99 we find from Table V that the upper confidence limit of the true dodder content is 40·5 per 10 kg. or 4·05 per 1 kg. If the agreed tolerance limit for true dodder content is, say, 5 dodder seeds per kg., then the tested stock will be considered as satisfactory. If, however, the agreed tolerance limit was only 1 dodder seed per kg., naturally the stock will be rejected. In Table IV we can find the upper bound of the probability of an error in this last judgment. This is lower than ·001. The Seller will probably not question this judgment and will not propose to increase the sample. However, if the tolerance limit is just below the confidence limit of the true dodder content, for instance say, 3·5, then perhaps it would be worth while to repeat the sampling and base the final verdict on larger data.

It will be seen that in the procedure which has been recommended above there are several points which are not completely specified. We deliberately refrain from propounding any definite rules. For instance, such questions as fixing the tolerance limits for the true mean dodder content and those of the confidence coefficients with which the seed testing stations should work, as well as the question of when the fit of the Poisson Series to the distribution of dodder seeds in samples from a number of sacks should be considered as showing heterogeneity of the material, etc., these should be answered by the Dodder Committee of the International Seed Testing Association, assisted by representatives of the seed traders. We think, however, that the work of this Committee should follow the methods indicated in the present paper. Among other matters it will be necessary to give an adequate definition to the term "dodder free" and to revise the meaning of the term "absolutely dodder free." We hope that the results presented above prove sufficiently that the present use of these terms may often be misleading.

I should like to add a word of thanks to Dr J. Neyman for his kind assistance.

A STUDY OF PREDYNASTIC EGYPTIAN SKULLS FROM BADARI BASED ON MEASUREMENTS TAKEN BY MISS B. N. STOESSIGER AND PROFESSOR D. E. DERRY.

By G. M. MORANT, D.Sc.

1. *Introduction.* In the season 1924—25 an expedition of the British School of Archaeology in Egypt, under the direction of Sir Flinders Petrie, excavated predynastic sites in the neighbourhood of Badari, 30 miles south of Asyut. The remains found could be assigned to an earlier date than any previously discovered belonging to the same era. Fifty-nine skulls were preserved and the majority of these were measured by Professor Derry in Cairo. They were subsequently sent by Sir Flinders Petrie to Professor Karl Pearson, then Director of the former Biometric Laboratory at University College, London, and the study of them by Miss Stoessiger (now Mrs Clapham) was published in 1927*. In the seasons 1928—30 Mr Brunton, who in 1924 had worked at Badari, leading a British Museum expedition, excavated more human remains associated with the Badari culture. A few of these came from Badari, but most are from a neighbouring site near Mostagedda. Eighty-three of the skulls were sent to Cairo and measured there by Professor Derry. He kindly placed his measurements of both series at the disposal of the present writer and those of the 1928—30 series are given in the appended tables. The objects of this paper are to compare the descriptive data provided by the two observers and to estimate from them the nature and racial affinities of the population represented by the two series of crania.

2. *A Comparison of Measurements of the* 1924—25 *Badari Series taken by Miss Stoessiger and Professor Derry.* In Miss Stoessiger's paper individual measurements are given of 59 skulls from Badari and unpublished measurements taken by Professor Derry are available for 53 of these. We thus have an unusual opportunity of comparing readings on the same specimens obtained by two observers working in different laboratories and following different techniques. Miss Stoessiger used the biometric technique without modification, and Professor Derry adopted that of the Monaco Congress† with some modifications and additions which he explained to the present writer. The measurements which are defined in these two schemes in ways which may be expected to give closely similar or identical results are the

* "A Study of the Badarian Crania recently excavated by the British School of Archaeology in Egypt." *Biometrika*, Vol. XIX. pp. 110—150.

† He refers to the English translation of the Monaco report published by Dr W. L. H. Duckworth in 1913 as: *International Agreements for the Unification* (*a*) *of Craniometric and Cephalometric Measurements*, (*b*) *of Anthropometric Measurements to be made on the Living Subject* (Cambridge University Press).

following (the biometric index letters being given first in the brackets, followed by the numbers of the Monaco report):

(i) Maximum calvarial length from the glabella in the median sagittal plane (*L*, 1).

(ii) Maximum calvarial breadth (*B*, 3). The two definitions differ since the biometric *B* is the maximum transverse diameter on the parietal bones, while the Monaco breadth is the maximum transverse whether found on the parietal bones or not. For Egyptian crania, however, the maximum breadth obtainable falls almost invariably on the parietal bones.

(iii) Minimum frontal breadth (*B'*, 5).

(iv) Basio-bregmatic height (*H'*, 4 *a*).

(v) Maximum horizontal circumference above the supra-orbital ridges and passing through the ophryon (*U*, 23 *a*).

(vi) Minimum arc from nasion to bregma (S_1).

(vii) Minimum arc from bregma to lambda (S_2).

(viii) Minimum arc from lambda to opisthion (S_3).

(ix) Total sagittal arc from nasion to opisthion (*S*, 22).

(x) Maximum bizygomatic breadth (*J*, 8).

(xi) Breadth between lowest points on zygomatico-maxillary sutures (*GB*).

(xii) Chord from nasion to basion (*LB*, 9).

(xiii) Chord from nasion to alveolar point (*G'H*, 12).

(xiv) Nasal height from nasion to point furthest removed from it on the margin of the left pyriform aperture (*NH*, *L*, Professor Derry took this—the Frankfurt—nasal height and not the Monaco measurement 13).

(xv) Maximum breadth of the pyriform aperture (*NB*, 14).

(xvi) Breadth of right orbit from the dacryon (O_1', *R*, 16). Professor Derry writes: "Where the lacrymal bone is broken, or missing, I use the angle formed by the union of frontal and frontal process of the maxilla, which I believe is more easily determined, and therefore more exact, than many other so-called 'points' on a skull." The biometric practice is to omit the measurement if the position of the dacryon is uncertain owing to defect of the lacrymal bone. Professor Derry gives the orbital breadth on both sides, and Miss Stoessiger that on the right side only.

(xvii) Maximum heights of orbits perpendicular to their breadths (O_2, *R* and O_2, *L*, 17).

(xviii) Profile angle, i.e. the angle between the line joining nasion to alveolar point and the Frankfurt horizontal plane (*P*∠).

(xix) Bicondylar breadth of the mandible (*w*, 25).

(xx) Symphyseal height of the mandible, i.e. the maximum chord from the tip of the process between the central incisors (intradental) to the lower border of the bone in the symphyseal plane (h_1, 29).

(xxi) Minimum breadth of the left ascending ramus of the mandible (*rb'*, 28 *a*).

It is not suggested that the biometric and Monaco definitions of all the above measurements accord exactly, and, indeed, those of the latter scheme are not full enough in many cases to decide this point at all definitely. It may be hoped, however, that the two sets of readings will be found in close accordance for these 21 characters. Agreement would not be expected in the case of the following measurements owing to differences in definition.

(xxii) Auricular height. Professor Derry writes: "I take this with the skull on the craniophor and with the lower margin of the left orbit on the same horizontal plane as the upper margin of the left auditory meatus. At times this is the same as the upper surface of the supports in the meatus, but more often it is above these owing to the slope of the bony roof of the meatus. In any case I place the point of the scriber against the margin of the meatus and then move it to the orbit and rotate the skull until the lower edge is at the same level. The sliding arm on the craniophor scale is then brought down until it just touches the skull lightly." The biometric practice is to make contact between the ear-rods of the craniophor and the poria in all cases and hence measurements obtained in the two ways cannot be expected to agree exactly. For the six comparisons which can be made Professor Derry's reading is greater than Miss Stoessiger's *OH* in all cases and the differences are between 1·4 and 3·2 mm.

(xxiii) Transverse cranial arc between "the most prominent point on the posterior root of the left zygoma, exactly above the auditory aperture" and the same point on the right passing through the bregma (23). This must always be considerably less than the biometric vertical arc from porion to porion (*Q'*) and in the case of five skulls for which both are available the differences range from 11·5 to 18·0.

(xxiv) Chord from basion to prosthion (10). This must be greater, almost invariably, than the chord from basion to alveolar point, defined to be the lowest point on the process between the central incisors (*GL*). Comparisons between the two measurements can be made in the case of 47 skulls: 41 show differences of the sign expected (the greatest being 3·5), two show zero differences and the remaining four differences of the opposite sign (the greatest being 1·2).

(xxv) Interorbital breadth. Professor Derry writes: "This is the actual width between the medial orbital walls when the two arms of the callipers are placed within the orbits and in contact with the medial walls." His definition differs from that of the Monaco technique (15) and from the biometric breadth between the dacrya (*DC*). Comparison with the latter can be made in six cases and the largest difference found is 1·3.

(xxvi) Bigonial breadth (26). According to the Monaco definition this measurement is taken on the external surface of the jaw between the bilateral points "at which the angle is formed between the lower margin of the body of the

jaw and the posterior border of its ascending ramus." This is practically the same definition as that of the biometric measurement w_2 which has been found to give unsatisfactory results in practice. The three differences (D. – S.) available are – 1·0, – 1·2 and + 2·3.

(xxvii) Maximum breadth of ramus (28 *b*). There is no biometric measurement similar to this.

(xxviii) Mandibular angle (32). This and the two following mandibular measurements were taken by Professor Derry on a mandible board, which he has described*, and which is precisely similar in construction to the instrument used by Miss Stoessiger. But the adjustment of the bone on the board was not the same in the two cases and hence the measurements are not comparable. The few differences available confirm this supposition. Professor Derry's mandibular angle is similar to the biometric $M\angle$.

(xxix) Projective length of corpus: similar to biometric $c_p l$.

(xxx) Projective length of ramus: similar to biometric rl.

The differences between the readings of the two observers for the same skulls may now be examined, in the case of the first 21 of the above measurements, to test whether the agreement is sufficiently close to be of no account in making comparisons with other series. Data are given in Table I for combined male and female specimens. Professor Derry took eight of the measurements in question on a few only of the Badari 1924—25 skulls which are sufficiently complete to provide readings for them. The information regarding personal equation available for these eight measurements in the table is obviously inadequate, fewer than 10 differences being given for each. As far as they go, however, the constants indicate that there is a satisfactory agreement for all except $P\angle$ and w_1. The definitions of the profile angle used by the two observers were the same, but there is reason to believe (see under Auricular height, measurement (xxii) above) that the adjustment of the skull in the craniophor was not made in the same way. This probably accounts for the rather large differences found and it will be safest to conclude that Professor Derry's profile angles are not comparable with Miss Stoessiger's. The same conclusion must be accepted in the case of the bicondylar breadth of the mandible (w_1). Much larger numbers of differences are available for the remaining 14 measurements—counting O_2, R and O_2, L as different measurements—in Table I. Several of the maximum individual differences in column 3 are considerably larger than those usually found between readings given by two workers in the same laboratory, and only four of the mean differences in the next column can be considered to differ insignificantly from zero (i.e. by less than three times their probable errors). The absolute sizes of the mean differences are of more importance, however, than the significance of their divergences from zero. They might be usefully compared with the probable errors of the means for the series, but these are as yet unknown. An approximation to them can be made by supposing that

* "A New Mylometer," *Journal of Anatomy and Physiology*, Vol. XLVIII. (1914), pp. 430—431.

the standard deviations for the Badari male skulls are the same as for longest Egyptian series available, viz. the *E* series of 26th—30th Dynasty skulls from Gizeh described by Professor Karl Pearson and Miss A. G. Davin*. The male Egyptian *E* standard deviations are in column 6 of the table and by using them the presumed probable errors of means for 35 Badari skulls are obtained and these are given in the last column. The numbers on which Miss Stoessiger's male means of cranial measurements of the Badari series are based actually range from 32 to 36. A comparison of the constants in columns 4 and 7 of Table I shows

TABLE I.

Data relating to the Differences between Miss Stoessiger's (S.) and Professor Derry's (D.) Measurements of the 1924—25 *Badari skulls.*

Character	No. of comparisons	Maximum difference (D. – S.)	Mean difference (D. – S.)	Standard deviation of differences	Male σ's of Egyptian *E* series	Presumed probable error of mean for 35 skulls
L	53	−2·0	−·32 ± ·060	0·65 ± ·043	5·72 ± ·09	·65
B	52	−3·5	+·02 ± ·068	0·73 ± ·048	4·76 ± ·08	·54
B′	53	+2·5	+·40 ± ·067	0·72 ± ·047	4·05 ± ·06	·46
H′	52	+3·0	−·34 ± ·065	0·70 ± ·046	5·03 ± ·08†	·57
U	53	+6·5	+1·29 ± ·200	2·16 ± ·142	13·77 ± ·22	1·57
S_1	8	−1·0	−·06	—	—	—
S_2	8	+ & −2·0	+·13	—	—	—
S_3	8	−2·0	−·37	—	—	—
S	8	−2·5	−·31	—	—	—
J	33	+2·5	−·03 ± ·087	0·74 ± ·061	4·57 ± ·08	·52
GB	45	−5·4	−·43 ± ·145	1·44 ± ·102	4·67 ± ·08	·53
LB	53	+ & −1·0	−·15 ± ·036	0·39 ± ·026	3·97 ± ·06	·45
G′H	47	−2·5	−·79 ± ·064	0·65 ± ·045	4·15 ± ·07	·47
NH, *L*	47	−2·3	−·64 ± ·064	0·65 ± ·045	2·92 ± ·05	·33
NB	47	−2·1	+·07 ± ·053	0·54 ± ·038	1·77 ± ·03	·20
O_1', *R*	46	−2·9	−·76 ± ·075	0·75 ± ·053	1·67 ± ·03‡	·19
O_2, *R*	45	+3·2	−·23 ± ·071	0·71 ± ·050	1·91 ± ·03	·22
O_2, *L*	44	−1·4	−·49 ± ·050	0·49 ± ·035	1·88 ± ·03	·21
P ∠	6	−2°·5	−°·58	—	—	—
w_1	3	+2·8	+1·37	—	—	—
h_1	2	−0·4	−·10	—	—	—
rb′	5	+0·2	+·02	—	—	—

that the ratio of the mean difference between the readings of the two observers to the presumed probable error of the male mean for the series ranges from ·04 to 4·0 (!). If the former constant is less than 0·7 of the latter the effect of personal equation on mean values may be considered negligible, and this is so for *L*, *B*, *H′*, *J*, *LB* and *NB*; if the ratio is between 0·7 and 1·4 (*B′*, *U*, *GB* and O_2, *R*) the "errors" will affect the comparisons appreciably but probably not render them

* "On the Biometric Constants of the Human Skull." *Biometrika*, Vol. XVI. (1924), pp. 328—363.

† For the vertical height from the basion (*H*) in place of the basio-bregmatic height (*H′*).

‡ For the "curvature" (O_1', *R*) in place of the dacryal (O_1', *R*) orbital breadth.

misleading; if the ratio is greater than 1·4 ($G'H$, NH, L, O_1', R and O_2, L) it may be supposed quite unsafe to neglect the consideration of personal equation in comparing means given by the two observers for a series made up by approximately 35 specimens. As measurements of variability will also be wanted for the series, it is necessary to consider the standard deviations, as well as the means, of the distributions of differences between the two sets of readings for the same skulls. These are given in Column 5 of Table I and they may be compared with the Egyptian E male standard deviations in the following column. If the former constant is less than 0·2 of the latter in the case of a particular character the agreement may be supposed satisfactory, and this is so for L, B, B', H', U, J, LB and $G'H$; if the ratio is between 0·2 and 0·4 (GB, NH, L, NB, O_2, R and O_2, L) the agreement is less satisfactory, and if greater than 0·4 (O_1', R) decidedly unsatisfactory. By applying these arbitrary tests to the means and standard deviations of the differences between Miss Stoessiger's and Professor Derry's readings on the same skulls, it is possible to group the measurements for which they used similar or identical definitions in the following way:

(*a*) those for which agreement is sufficiently close to make it possible to neglect the effect of personal equation—L, B, H', J, LB, and probably S_1, S_2, S_3, S, h_1 and rb', though few data for these last six from which estimates of their accuracy can be obtained are available;

(*b*) those for which agreement is less satisfactory, so that there is a danger of personal equation affecting comparisons between constants for series appreciably, though not to a marked extent—B', U, GB, NB, O_2, R;

(*c*) those for which comparisons between the constants will be definitely invalid owing to personal equation—$G'H$, NH, L, O_1', R, and O_2, L, and probably $P\angle$ and w_1, though few data for these last two are available.

There can be no doubt that personal equation would be of much less account if measurements of two workers in the former Biometric Laboratory were being compared than in the case of the present comparison, but this might have been anticipated.

3. *The Sexing of the Badari Series.* Three estimates of the sexes of the adult 1924—25 Badari skulls are available. The first of these to be published is that of Miss Stoessiger's paper in which it is said that the specimens were sexed by Professor Karl Pearson and the present writer. There are 36 males and 22 females distinguished, and 53 of these have been sexed by Professor Derry. He is willing to accept the determination given in the paper in *Biometrika* in 47 cases, but differences must be recorded for the remaining six which are: Nos. 5394, S. ♂?, D. ♀; 5435, S. ♂, D. ♀; 5438, S. ♂, D. ♀; 5743, S. ♂, D. ♀?; 6004, S. ♂, D. ♀; 5383, S. ♀, D. ♂. The third sexing of the series is that given in the report on the excavations* from which the following is quoted (pp. 19—20):

* *The Badarian Civilisation and Predynastic Remains near Badari.* By Guy Brunton and Gertrude Caton-Thompson, British School of Archaeology in Egypt and Egyptian Research Account, Thirtieth Year, 1924. London, 1928.

"There was no sign that the bodies had been preserved in any way. The skin was sometimes still visible, occasionally well preserved, but the internal organs had always disappeared....As regards segregation of sexes in cemeteries, a curious fact emerges. Female graves were not placed apart; where there are graves of women, there are also graves of men alongside. Certain areas, however, were reserved for men....The most easily observed feature was the hair....It is interesting to note that in no case could a beard or moustache be detected."

The method of sexing used is not described, but it appears from the remarks on individual skeletons that the length and condition of the hair was taken as one criterion, and the nature of the grave furniture also gave some guidance presumably. Sexes are given in the report (A.R.) for 42 of the skulls sexed both in Miss Stoessiger's paper (S.) and by Professor Derry (D.). It is found that for 24 of these, all three estimates agree; for 30, A.R. agrees with S.; for 35, A.R. agrees with D., and for seven A.R. differs from both S. and D. The last seven are Nos. 5411, A.R. ♀ "long wavy black hair arranged in twisted tresses"; 5745, A.R. ♀ "hair thick, black and wavy, bracelet"; 5815, A.R. ♀ "dark, slightly wavy hair"; 5359, A.R. ♂; 5360, A.R. ♂; 5368, A.R. ♂; 5370, A.R. ♂ "in male cemetery." A re-examination of these seven skulls suggests that an anatomist would have little hesitation in assigning sexes to them opposite to those given in the report, in spite of the fact that the sexing of the Badari skulls is particularly difficult owing to the fact that the type is a feeble one showing little muscular development. It is unfortunate that there is no closer agreement between the three estimates, and the effect of the discordance on means must at least be examined. Of the measurements which have been found to be least affected by personal equation, the three major calvarial diameters are the ones most suitable to use for this purpose and means for them are given in the following table, computed only for the 42 skulls for which all three estimates of sex are available and using Miss Stoessiger's measurements in every case.

Sexing	Sex	*L*	*B*	*H'**
Stoessiger	♂	183·4 (27)	131·2 (27)	134·0 (26)
Derry		183·5 (23)	131·9 (23)	134·5 (22)
Archaeological Report		182·2 (24)	131·6 (24)	133·6 (23)
Stoessiger	♀	177·2 (15)	131·1 (15)	128·9 (15)
Derry		178·3 (19)	130·2 (19)	129·4 (19)
Archaeological Report		179·7 (18)	130·6 (18)	130·3 (18)

The means are seen to be in better accordance than might have been anticipated. For *L* and *H'* those given by Miss Stoessiger and Professor Derry agree better than either set does with those based on the sexing in the Archaeological

* *H'* is not given for one skull which is male according to all three estimates.

Report, but for *B* the last two are in closest agreement. In the case of two of the estimates, a similar comparison can be made based on 53 skulls, and the means are:

Sexing	Sex	*L*	*B**	*H'*†
Stoessiger	♂	182·2 (33)	130·9 (33)	133·1 (32)
Derry ...		182·2 (29)	131·5 (29)	133·4 (28)
Stoessiger	♀	177·0 (20)	130·2 (19)	128·9 (20)
Derry ...		177·9 (24)	129·6 (23)	129·3 (24)

The pairs of means here are actually all closer together than the corresponding ones based on the 42 skulls. We are comparing two anatomical estimates of sex which almost certainly differ owing to the interchange of a few doubtfully male or doubtfully female specimens, and the way in which these are assigned does not appear to influence the means greatly. But still the greatest observed differences between the means are of an order which would have indicated unreliable measurements if they had been found as mean differences between the readings given by two different observers (cf. data in Table I). The personal equation in sexing is hence a factor which is likely to affect comparisons between series appreciably. The sexes given in Miss Stoessiger's paper were accepted in making the comparisons in the following section of the present paper since they are given for all the specimens, while the other two records only give sexes for parts of the total 1924—25 series. It is not suggested that the former sexing is more accurate than the others when available. For the 1928—30 series, we have only Professor Derry's determinations to follow.

4. *A Comparison of the* 1924—25 *and* 1928—30 *Badari Series.* Constants are given in Table II based on Miss Stoessiger's measurements of the 1924—25 series and Professor Derry's of the 1928—30 series. The data there only relate to those characters for which it has been shown that comparisons between the measurements of the two observers are not likely to be vitiated to a marked extent owing to their personal equations‡. Standard deviations have not been calculated for the series excavated on the two occasions, since it is clear that their means are in close agreement, but a non-metrical comparison of their distributions suggests that they have approximately the same variability as one another and as the combined sample in the case of every character. The standard deviations of the last were calculated and the probable errors of the means are given (= p.e. M_p, say). The probable errors of the means of the two component samples will certainly

* *B* is not given for one skull which is female according to both estimates.

† *H'* is not given for one skull which is male according to both estimates.

‡ All such are included in Table II (see p. 302 below) except the height of the right orbit (O_2, R). As the height of the left orbit was found to be an unreliable measurement, it appeared safer to class O_2, R with those of that kind and data for it are given in Table III.

be larger than these, but their differences may be compared with $\sqrt{2 \times (\text{p.e. } M_p)^2}$ to give an estimate of whether they are significant or not. The difference divided by this quantity is greater than 3 in the case of the male $100\,H'/L$ (3·4), H' (3·6) and J (3·9), and the female S (3·1), S_2 (3·3), NB (3·8), $100\,B/H'$ (3·9), GB (4·5), B (4·7), $100\,H'/L$ (4·7) and L (4·9). The ratio is greatest for the last comparison, and if it is assumed that for this the standard deviations of the component distributions are the same as for the pooled distribution—which is a reasonable assumption—then the female mean L for the 1924—25 series will be $176{\cdot}7 \pm {\cdot}52$ and for the 1928—30 $174{\cdot}4 \pm {\cdot}43$. The difference between these is 3·4 times its probable error. It will be safe to suppose that the differences between the mean measurements of the two separate series are all insignificant, except possibly in the case of the female values for L, B, GB and $100\,H'/L$, and no clearly significant differences are found for these. The conclusion that the Badari series excavated on the two occasions represent the same racial type appears to be sufficiently justified and hence they have been pooled.

This conclusion might be invalidated if clearly significant differences were found for any characters not in Table II. Table III gives the means for those characters for which identical or very similar definitions were used by the two observers, but for which their readings on the same skulls were found to differ appreciably or markedly. It will not be legitimate to pool data for these characters. The "divergences expected" in the last column of the table are those actually found (see Table I) between the mean measurements of the two observers for the same skulls. The constants for the 1924—25 and 1928—30 series in Table III are still in close agreement, and it will be safe to assume that none indicate possible differentiation of the types—paying due regard to the divergences expected owing to personal equation in comparing the means—except possibly in the case of the upper facial ($G'H$) and nasal (NH, L) heights and indices, and in the case of the bicondylar breadth of the mandible (w_1). The divergence expected in the case of the last character is only based on the comparison of measurements of three specimens, and hence it is quite likely to be misleading, and Miss Stoessiger's and Professor Derry's readings may be directly comparable. But the measures of personal equation in the case of the other characters which are showing larger differences than would have been anticipated are based on adequate numbers of comparisons. The standard deviations were calculated for the two series separately giving the results provided in Table III, p. 303, the means for the 1928—30 series being given values which they would presumably have had if measured by Miss Stoessiger.

The difference between the male means for $G'H$ is here 3·4 times its probable error and the difference between the female means for NH, L is 3·8 times its probable error, while the other two differences are quite insignificant. It appears that the 1928—30 type had a significantly higher face than the 1924—25 type, though part of the observed difference may possibly be due to the fact that the series were sexed by different observers. The discordance is not sufficiently marked

TABLE II.

Constants from Miss Stoessiger's (S.) and Professor Derry's (D.) Measurements of different Badari Series.

Measurements		Means						Variabilities: pooled (S. & D.)			
		♂			♀			Standard Deviations		Coefficients of Variation	
		1924—25 (S.)	1928—30 (D.)	Pooled (S. & D.)	1924—25 (S.)	1928—30 (D.)	Pooled (S. & D.)	♂	♀	♂	♀
Least affected by personal equation	L	182·3 (36)	182·6 (45)	182·5 ± ·44 (81)	176·7 (22)	174·4 (33)	175·3 ± ·33 (55)	5·88 ± ·31	3·65 ± ·23	3·22 ± ·17	2·08 ± ·13
	B	130·8 (36)	132·2 (45)	131·6 ± ·41 (81)	130·3 (21)	127·6 (33)	128·7 ± ·41 (54)	5·47 ± ·29	4·47 ± ·29	4·16 ± ·22	3·47 ± ·23
	H'	132·9 (34)	135·1 (43)	134·1 ± ·43 (77)	129·1 (22)	129·9 (33)	129·6 ± ·37 (55)	5·57 ± ·30	4·08 ± ·26	4·16 ± ·23	3·15 ± ·20
	LB	99·3 (35)	101·2 (41)	100·3 ± ·34 (76)	96·1 (22)	96·5 (33)	96·3 ± ·31 (55)	4·42 ± ·24	3·41 ± ·22	4·40 ± ·24	3·54 ± ·23
	S_1	127·0 (35)	126·9 (44)	126·9 ± ·42 (79)	123·2 (22)	122·3 (32)	122·7 ± ·37 (54)	5·52 ± ·30	4·04 ± ·26	4·35 ± ·23	3·29 ± ·21
	S_2	129·1 (35)	130·0 (44)	129·6 ± ·50 (79)	128·5 (22)	125·2 (32)	126·5 ± ·70 (54)	6·64 ± ·36	7·68 ± ·50	5·13 ± ·28	6·07 ± ·40
	S_3	115·7 (34)	113·9 (44)	114·7 ± ·54 (78)	111·7 (22)	112·3 (32)	112·1 ± ·48 (54)	7·09 ± ·38	5·22 ± ·34	6·18 ± ·34	4·66 ± ·30
	S	372·0 (35)	370·8 (44)	371·3 ± ·90 (79)	363·4 (22)	359·8 (32)	361·3 ± ·82 (54)	11·88 ± ·64	8·95 ± ·58	3·20 ± ·17	2·48 ± ·16
	J	122·5 (32)	124·9 (29)	123·6 ± ·44 (61)	117·7 (13)	118·0 (26)	117·9 ± ·50 (39)	5·09 ± ·31	4·67 ± ·36	4·12 ± ·25	3·96 ± ·30
	100 B/L	71·8 (36)	72·4 (45)	72·1 ± ·24 (81)	73·8 (21)	73·2 (33)	73·4 ± ·27 (54)	3·17 ± ·17	2·89 ± ·19	—	—
	100 H'/L	73·1 (34)	74·1 (43)	73·7 ± ·21 (77)	73·1 (22)	74·5 (33)	73·9 ± ·21 (55)	2·77 ± ·15	2·33 ± ·15	—	—
	100 B/H'	98·3 (34)	98·2 (43)	98·2 ± ·41 (77)	101·0 (21)	98·4 (33)	99·4 ± ·47 (54)	5·30 ± ·29	5·09 ± ·33	—	—
	h_1	32·6 (34)	32·5 (38)	32·5 ± ·24 (72)	31·5 (18)	30·8 (29)	31·1 ± ·22 (47)	3·01 ± ·17	2·25 ± ·16	9·26 ± ·52	7·23 ± ·51
	rb'	33·6 (39)	34·2 (40)	33·9 ± ·17 (79)	32·0 (23)	32·3 (33)	32·2 ± ·21 (56)	2·28 ± ·12	2·37 ± ·15	6·72 ± ·36	7·36 ± ·47
More affected by personal equation	B'	91·1 (36)	91·1 (45)	91·1 ± ·29 (81)	89·4 (22)	88·9 (34)	89·1 ± ·37 (56)	3·92 ± ·21	4·16 ± ·27	4·30 ± ·23	4·67 ± ·30
	U	501·3 (36)	499·0 (44)	500·0 ± 1·11 (80)	489·0 (22)	485·6 (33)	487·0 ± ·81 (55)	14·74 ± ·79	8·90 ± ·57	2·95 ± ·16	1·83 ± ·12
	GB	94·6 (34)	94·4 (42)	94·5 ± ·39 (76)	89·6 (18)	9[illegible]·4 (33)	91·4 ± ·44 (51)	4·98 ± ·27	4·62 ± ·31	5·27 ± ·29	5·06 ± ·34
	NB	24·9 (34)	24·8 (42)	24·8 ± ·14 (76)	23·6 (20)	24·3 (34)	24·0 ± ·13 (54)	1·80 ± ·10	1·43 ± ·09	7·26 ± ·40	5·96 ± ·39

Character	Means				Standard Deviations			
	♂		♀		♂		♀	
	1924—25	1928—30	1924—25	1928—30	1924—25	1928—30	1924—25	1928—30
G'H	67·1 ± ·46 (34)	[69·3] ± ·45 (43)	64·8 ± ·63 (20)	[66·0] ± ·33 (34)	3·95 ± ·32	4·41 ± ·32	4·18 ± ·45	2·83 ± ·23
NH, L	48·4 ± ·32 (34)	[49·4] ± ·36 (43)	45·6 ± ·42 (20)	[47·4] ± ·23 (34)	2·78 ± ·23	3·47 ± ·25	2·81 ± ·30	1·99 ± ·16

TABLE III.

Mean Measurements of two Badari series, from Readings by Miss Stoessiger (S.) and Professor Derry (D.), between which Agreement is not expected owing to Personal Equation.

Character	♂		♀		Divergence expected (S. – D.)
	1924—25 (S.)	1928—30 (D.)	1924—25 (S.)	1928—30 (D.)	
G'H	67·1 (34)	68·5 (43)	64·8 (20)	65·2 (34)	+·8
NH, L	48·4 (34)	48·8 (43)	45·6 (20)	46·8 (34)	+·6
O_1', R	38·4 (33)	38·3 (38)	37·6 (20)	37·2 (32)	+·8
O_1', L	—	37·4 (37)	—	36·3 (32)	—
O_2, R	32·0 (34)	31·5 (38)	31·3 (21)	32·2 (32)	+·2
O_2, L	32·1 (33)	31·6 (37)	31·4 (18)	31·9 (33)	+·5
P ∠	84°·0 (33)	82°·2 (41)	83°·3 (20)	81°·2 (33)	+0°·6 ?
100 *G'H/GB*	70·9 (34)	72·5 (41)	71·7 (17)	70·5 (33)	—
100 *NB/NH, L*	51·6 (34)	50·9 (42)	51·8 (20)	51·9 (34)	—
100 $O_2/O_1', R$	83·3 (33)	82·2 (37)	83·4 (20)	86·6 (32)	—
100 $O_2/O_1', L$	—	84·7 (37)	—	87·8 (32)	—
w_1	109·5 (30)	109·9 (36)	105·2 (16)	104·1 (30)	– 1·4 ?

to render the pooling of the series unjustifiable. Table IV gives the means for the remaining measurements taken by Professor Derry on the 1928—30 series and for similar, but differently defined, measurements taken on the 1924—25 series by Miss Stoessiger. It is clear that entirely fallacious conclusions would have been reached if some of these corresponding pairs had been supposed comparable.

5. *The Nature of the Badari Series.* Certain features of the Badari series can be estimated from the data given in Tables II—IV above. The characters for which pooled means are given (Table II) have constants based on fairly adequate numbers of skulls and the sex ratios (male mean/female mean) for the absolute measurements are found to be unusually small, though of the same order as those

TABLE IV.

Mean Measurements of two Badari Series for corresponding but differently defined Characters.

Prof. Derry's measurements	1928—30 (Derry)		Corresponding but differently defined biometric measurements	1924—25 (Stoessiger)	
	♂	♀		♂	♀
Monaco transverse arc ...	296·1 (41)	283·0 (32)	Q	302·0 (34)	298·8 (20)
Auricular height ...	115·7 (41)	111·8 (33)	OH	111·0 (34)	108·6 (21)
Chord bas.-prosth. ...	98·5 (40)	94·2 (33)	GL	95·0 (33)	92·6 (20)
Interorbital breadth ...	21·7 (40)	21·1 (32)	DC	22·4 (32)	20·6 (21)
100 Chord bas.-prosth./LB	97·4 (40)	97·7 (33)	—	—	—
Bigonial breadth ...	91·6 (40)	85·8 (31)	w_2	88·8 (32)	86·2 (21)
Max. breadth of ramus ...	42·4 (39)	40·2 (33)	—	—	—
Length of corpus ...	79·9 (40)	75·3 (33)	$c_p l$	76·2 (33)	74·1 (19)
Length of ramus... ...	61·1 (38)	55·0 (33)	rl	57·6 (33)	53·2 (18)
Mandibular angle ...	117°·7 (41)	121°·0 (33)	$M\angle$	120°·0 (34)	123°·3 (19)

observed for other series*. Owing to the difficulty of sexing the Badari series, it would be unwise to attach much importance to this result. The mean cephalic and breadth-height indices are about one unit less for the male than for the female series, as is usually found. Other expected relations are that the female orbital index (Table III) and mandibular angle (Table IV) are greater than the male values. It is probable that the differences between the means for the two sexes in the case of all the other indices and of the profile angle are quite insignificant. There is every reason to believe that the male and female series represent the same racial type.

Sexual comparisons in variability can only be made in the case of the 18 characters dealt with in Table II. If coefficients of variation are used for the absolute measurements and standard deviations for the indices, it is found that for 15 of these the male variability is the greater. Dividing the difference of the constants by its probable error gives values over 3 in the case of S (3·1), H' (3·3), S_1 (3·4), S_3 (3·4), L (5·3) and U (5·6), while for all of these it is the male variability which is the greater. A similar clear preponderance of male over female variability has been observed in the case of the long Egyptian E series of the 26th—30th Dynasty skulls, but for other Egyptian and non-Egyptian series it has been far more usual to find a closer approach to sexual equality in this respect. The material suggests that there are racial differences between sexual differences in variability and the Badari type appears to be close to one extreme of the range. It is possible that inaccuracies in sexing have favoured the last conclusion.

* Sex ratios for seven calvarial measurements are given in *Biometrika*, Vol. xxv. (1933), p. 259 in the case of the Kerma, the E Egyptian and the Teita negro series. The Badari values for each character are smaller than any shown there, though the difference between the Badari and the next smallest sex ratio is small in every case, its maximum value being ·009.

A comparison of the variability of the Badari and other series is of particular interest since it is believed that we are dealing with the earliest cranial sample of any length which has been obtained from any part of the world*. It is implied, or distinctly stated, in a good deal of the literature of physical anthropology that present-day populations are racially far more heterogeneous, and hence far more variable, than those which existed at the beginning of the Christian era, say, and that by going back further in time we must approach closer and closer to that ideal and undefined population which may be called a "pure" race. There appears to be no valid reason for attempting to divide the Badari series into sub-samples which would show lesser variabilities than the total sample. It is reasonable to assume that the people represented formed part of an intermarrying group which could not be divided into racial components owing to their complete inter-mixture for several generations. The biometrician accepts such a group as representing a racially homogeneous type, and comparisons may be made with other samples presumed to have been drawn at random from similarly constituted populations. The Badari variabilities for the 16 cranial characters in Table II (omitting the mandibular measurements h_1 and rb') are compared below with those of the long Egyptian *E* series, using coefficients of variation for the absolute measurements and standard deviations for the indices:

Male.—Badari variabilities greater 11 characters, Egyptian *E* greater 5; differences exceeding three times their probable errors *B* (3·2, Badari greater).

Female.—Badari variabilities greater 7 characters, Egyptian *E* greater 9; differences exceeding three times their probable errors 100 H'/L (3·2), L (4·2), S_3 (4·5), U (4·8), S_1 (5·6), the Egyptian *E* being the greater in these five cases.

It appears that the male Badari series is slightly more variable than the male Egyptian *E*, but for the females the position is reversed and the difference is more significant. The Egyptian *E* has been compared with several other cranial series in a similar way, and it has been found that its variability is slightly less than those of samples from different parts of the world which are presumed to represent racially homogeneous populations. The following comparison is between the Badari constants and those given for the Farringdon Street English series which was obtained from a single London cemetery used in the seventeenth century†, no selection of specimens on which measurements could be taken having been made:

Male.—Badari variabilities greater 4 characters, Farringdon Street greater 12; differences exceeding three times their probable errors *GB* (3·1, Farringdon Street greater).

* Since the later discovery of Tasian remains, the Badari is generally considered to be the second oldest culture of predynastic Egypt known. If the beginning of dynastic times is dated at 3000 B.C. and the length of the predynastic period is estimated as a minimum of 2000 years, then the Badari people must have lived as long ago as 4000 B.C. and probably earlier. Most authorities apparently agree in making them ante-date all European neolithic cultures by a considerable period. See, for example, Miles Burkitt and V. Gordon Childe: "A Chronological Table of Prehistory." *Antiquity*, Vol. VI. (1932), pp. 185—205.

† Beatrix G. E. Hooke, "A Third Study of the English Skull with special reference to the Farringdon Street Crania." *Biometrika*, Vol. XVIII. (1926), pp. 1—55.

Female.—Badari variabilities greater 2 characters, Farringdon Street greater 14; differences exceeding three times their probable errors LB (3·4), NB (3·6), S_3 (4·3), H' (4·8), S (5·3), 100 H'/L (5·6), S_1 (7·6), U (7·7), L (8·8), the Farringdon Street being the greater in these nine cases.

While there appears to be no marked difference in variability between the two male series, the Farringdon Street female must be considered to be very significantly more variable than the Badari female series. The last conclusion is likely to convey a false impression, however, since it may suggest that the one series is much more variable than the other. To obtain some measure of the order of the difference, the Farringdon Street constant was divided by the Badari constant in the case of each character, and then the average of these ratios was found for the 16 characters. This was also done in the case of the other three comparisons above and the results are: with Egyptian E, male 0·952, female 1·073; with Farringdon Street, male 1·080, female 1·281. Judging in this way, the English female variability is thus of the order 1·3 times the Badari female variability, which is an extraordinarily low ratio, since we are here dealing with series accepted as representing racially homogeneous types which show a divergence in variability approaching the maximum found among such series*. In the last 6000 years there appears to have been little change in the variability of racial populations, and it is safe to conclude, perhaps, that "pure" types much less variable than any found to-day have only existed in that period in the imaginations of some anthropologists.

6. *Racial Comparisons.* Mean cranial measurements based on 30 or more male specimens have been collected for 150 series representing racial types from different parts of the world, excluding many available for ancient Egypt and Europe to prevent these regions being over-represented. It is found that the Badari means are quite close to the inter-racial averages (i.e. the unweighted averages of the serial means), except in the case of: B which is small, though 12 out of 150 means are smaller; B' small, but 7 out of 126 smaller; J small, but 2 out of 141 smaller; U small, but 14 out of 116 smaller; 100 B/L low, but 17 out of 150 lower; 100 B/H' low, but 15 out of 148 lower; NH, L small, but 19 out of 144 smaller; O_2 small, but 4 out of 148 smaller (than Stoessiger's mean 31·5); S_2 large, but 17 out of 98 larger. The Badari type is thus seen to have calvarial and facial breadths which are unusually, but not extremely, small, while its calvarial antero-posterior lengths (apart from the S_2 arc from bregma to lambda) and heights are not at all exceptional. The horizontal circumference (U) is hence

* There appears to be only one cranial series, for which the constants have been published, which is appreciably less variable than the Badari and Egyptian E. This is one of Guanche skulls measured by Prof. E. A. Hooton ("The Ancient Inhabitants of the Canary Islands." *Harvard African Studies*, Vol. VII. 1925). Data for 12 of the 16 cranial characters in Table II above are given. Dividing the Tenerife constants—coefficients of variation for absolute measurements and standard deviations for indices—by the Badari values gives for these a mean male ratio 0·914 and a mean female 1·040. It is not surprising to find that the population of a small island shows peculiarly small variation since inbreeding must have been common. Most European samples—whether derived from single cemeteries or from several cemeteries believed to have been used by closely related groups of people—show variabilites which are close to the Farringdon Street values.

small and the cephalic index low. Capacities could not be found directly owing to the fragility of the skulls, but the average for them must be unusually small. The facial skeleton has unusually small heights. Comparative material for the mandible is scanty, but the Badari bicondylar breadth (w_1) appears to be the smallest male mean recorded, thus agreeing with the conclusion that the breadths of the cranium are unusually small. Comparisons between the Badari means for single characters and those available for other ancient Egyptian series have been made by Miss Stoessiger*, and it has been shown that the Badari type diverges slightly from them in being more prognathous and in having a higher nasal index. In these respects it is more negroid.

The racial relationships of the Badari type can be judged better from coefficients of racial likeness. There are 27 reduced values, of which several have been previously published, in Table V below and the male means given by Miss Stoessiger were used in computing all these constants. New means based on larger numbers of skulls are now available for 12 of the characters used in computing the coefficients, but these are all very close to Miss Stoessiger's means, and by using hers in all cases we have all 31 means based on approximately the same numbers which is an advantage. As would have been anticipated, the closest connections are found with other Predynastic Egyptian series. The dynastic series are further removed, and types more similar to the Badari than some of them are found in Abyssinia to-day and also among the criminal classes in Cairo (Sydney Smith). All other African types with which comparisons have been made are more widely removed from the Badari type. Miss Stoessiger found that there is an extraordinarily close similarity between the last and some Indian types of cranium, and the revised coefficients in Table V confirm her conclusion. A comparison of the reduced coefficients there with several hundred others, published and unpublished, suggests the following conclusions:

(i) The Badari type resembles most closely those of the other Predynastic Egyptian series available. Its connections with these are as close as those normally found between different samples representing the populations of the same country at different times, but in the same era. None are so close that the Badari and any other predynastic series can be supposed to represent precisely the same population†. The Badari resembles the other early predynastic series more closely than it does the late predynastic series. With advancing time the type in the predynastic era was changing by becoming less prognathous and by acquiring slightly lower nasal and higher cephalic indices. A slow progression in the same direction appears to have continued throughout the whole of the dynastic era and down to Roman times in Upper and Middle Egypt at least. The later of the known Egyptian types for this period differ quite markedly from the Badari

* *Loc. cit.* p. 124.

† This seems to be the safest conclusion to draw in spite of the fact that an insignificant coefficient of racial likeness is found between the Badari male means and those of the Early Predynastic series from Abydos, El-Amrah and Hou measured by Thomson and MacIver; only 14 of the 31 measurements used whenever possible in computing the coefficients are available for the latter.

TABLE V.

Male reduced Coefficients of Racial Likeness between the Badari and other Series.

Ancient Egyptian and related Series	C.R.L.*	Negro Series and Hottentots‖	C.R.L.	Other Series	C.R.L.
Early Predynastic: Abydos, El-Amrah and Hou†	1·72±·69 (14)	Hottentots	16·84±·69 (24)	Dravidians¶ ...	6·70±·69 (18)
Predynastic: Naqada A and Q†	4·02±·38 (31)	Negroes from Egypt ...	18·38±·54 (27)	Sardinians** ...	9·72±·59 (14)
Late predynastic: El-Amrah and Hou†	7·62±·49 (14)	Tanganyika	22·83±·57 (22)	Veddahs¶ ...	10·53±·63 (20)
12th—13th dynasty: Kerma‡	7·77±·33 (31)	Teita	24·54±·47 (31)	Nepalese¶ ...	12·73±·44 (31)
6th—12th dynasty: Denderah†	7·78±·45 (14)	Angoni	32·67±·71 (23)	Galla and Somali‖	14·26±·67 (22)
Modern Abyssinians: Tigre district†	8·45±·41 (28)	Gaboon	44·51±·40 (29)	Hindus¶ ...	14·54±·44 (21)
Modern Egyptians: Cairo§	9·83±·58 (15)	Congo	46·33±·44 (31)	Tamils¶ ...	14·64±·50 (31)
12th—15th dynasty: Hou and Abydos†	10·34±·57 (14)	Kaffirs	54·44±·45 (29)	Bushmen‖ ...	37·96±·53 (24)
18th—20th dynasty: Thebes†	15·15±·43 (29)	Cameroons	61·34±·37 (28)	Tasmanians‖ ...	49·55±·49 (24)

* The numbers in brackets are the numbers of characters on which the coefficients are based.
† Means in *Biometrika*, Vol. XVII. (1925), pp. 14—36.
‡ Means and coefficient given by Collett: *Ibid.* Vol. XXV. (1933), pp. 261 and 270.
§ Means given by Sidney Smith in *Journal of Anatomy*, Vol. LX. (1926), p. 123.
‖ Means or references and coefficients given by Kitson, *Biometrika*, Vol. XXIII. (1931), pp. 285—298.
¶ Means or references given by Woo and Morant, *Ibid.* Vol. XXIV. (1932), pp. 114—118.
** Means in *Ibid.* Vol. XX[B]. (1928), pp. 370—371.

Locality	Grave No.	Calvarial Measurements								
		L	*B*	*H'*	[*OH*]	*B'*	*U*	S_1	S_2	S_3
Mostagedda	28/209	187·0	131·0	141·5	119·5	91·5	515·0	125·0	134·0	116·0
,,	28/302	182·5	127·0	137·0	114·5	90·0	505·5	128·5	130·0	116·0
,,	28/447	175·0	129·5	129·0	114·0	95·5	499·0	134·0	128·0	108·0
,,	28/459	174·0	120·0	138·0	116·0	87·0	486·0	116·0	127·0	110·0
,,	28/461	184·0	127·0	135·5	116·5	96·0	—	125·0	133·0	112·0
,,	28/466	175·0	123·0	127·5	110·0	87·0	472·5	122·0	125·0	103·0
,,	28/471 A	176·5	137·0	127·0	112·5	91·0	494·0	116·0	136·0	107·0
,,	28/471 B	173·0	123·5	141·5	118·0	91·0	486·0	127·0	135·0	105·0?
,,	28/472 B	183·5	136·5	140·0	116·0	86·5	500·0	127·5	128·0	122·0
,,	28/473 B	180·0	122·0	137·0	111·0	92·0	490·0	122·0	119·0	106·0
,,	28/490	182·0	138·0	132·5	114·0	102·0	515·0	129·0	123·0	115·0
,,	28/496	181·0	142·0	134·5	115·0	93·0	510·0	128·0	107·0	129·0
,,	28/498	192·0	123·0	—	—	92·5	513·0	118·0	138·0	121·0
,,	28/1205	184·5	139·0	132·0	115·5	93·5	510·0	123·0	132·0	112·0
,,	28/1208	181·0	131·0	142·0	119·0	86·0	500·0	127·0	134·0	105·0
,,	28/1211	187·5	145·0	138·5	121·0	92·0	533·5	138·0	127·0	125·0
,,	28/1216	181·5	127·0	—	120·0	93·0	504·0	131·0	129·0	114·0
,,	28/1223	190·0	138·0	140·0	121·5	96·0	526·5	130·0	140·0	122·0
,,	28/1231	179·5	136·0	136·0	119·0	94·0	504·0	124·0	131·0	109·0
,,	28/1246	192·0	129·0	134·5	115·5	89·0	516·0	127·0	140·0	119·0
,,	28/2208	171·0	130·0	138·0	117·0	88·5	485·0	125·0	132·0	100·0
,,	28/2210	169·0	128·0	128·0	111·0	90·0	473·0	119·0	120·0	110·0
,,	28/2231	180·0	139·0	131·5	115·0	84·0	499·0	120·0	124·0	112·0
,,	28/2232	188·0	125·0	132·0	112·0	87·0	500·0	121·0	127·0	129·0
,,	28/2235	180·5	136·0	133·0	114·0	88·0	502·0	123·0	133·0	119·0
,,	28/2252	177·0	132·0	132·0	113·0	85·0	490·0	121·0	135·0	117·0
,,	28/2603	171·0	129·0	131·0	111·0	92·5	480·0	122·0	123·0	105·0
,,	28/2707	185·5	135·5	139·0	117·0	85·0	504·0	135·0	131·0	104·0
,,	29/3257	173·0	129·0	134·5	118·0	82·5	484·0	126·5	134·0	111·0
,,	29/3300	186·0	135·0	135·0	115·0	88·5	519·0	124·0	140·0	116·0
,,	29/3505	184·5	135·0	135·5	115·0	96·5	513·0	132·0	129·0	112·0
,,	29/3506	185·5	139·0	130·0	117·5	95·0	521·0	140·0	135·0	106·0
,,	29/3512	191·5	133·0	148·5?	123·0	91·5	518·0	140·0	147·0	108·0
,,	29/3526	178·0	127·0	131·5	—	95·5	493·0	119·0	123·0	115·0
,,	29/3532	179·5	131·5	133·0	117·0	89·0	495·0	129·0	132·0	99·0
,,	29/3534	188·5	131·0	136·0	116·0	95·0	515·0	130·0	137·0	111·0
,,	29/3539	178·0	130·5	129·0	112·0	88·0	493·0	122·0	123·0	112·0
,,	29/10031	182·5	133·5	131·0	110·0	90·5	504·0	126·0	121·0	119·0
Badari	30/2006	195·0	134·5	137·0	115·0	93·0	528·0	129·0	131·0	129·0
,,	30/2007	185·5	142·0	131·0	118·0	91·0	517·0	127·0	125·0	128·0
,,	30/2008	189·0	130·0	136·0	118·0	89·0	511·0	127·0	135·0	114·0
,,	30/2009	194·0	133·0	135·0	113·0	99·0	530·0	132·0	130·0	114·0
,,	30/2033	192·5	143·0	139·0	118·5	97·0	537·5	141·0	131·5	120·0
,,	30/2504	—	—	—	—	—	—	—	—	—
,,	30/2513	178·5	126·0	135·0	—	92·0	484·0	—	—	—
,,	30/2524	190·0	137·5	143·0?	—	89·0	520·0	135·0	126·0	126·0

* An index letter in square brackets

nts					Facial Measurements							
S_2	S_2	S	[*Broca's Q'*]	LB	[*Prosthion GL*]	[$G'H$]	GB	J	[DC]	[$O_1'R$]	[$O_1'L$]	[O_2R]
134·0	116·0	375·0	299·0	—	—	74·5	99·5	—	—	—	—	—
130·0	116·0	374·5	290·0	99·0	99·0	73·5	91·5	—	23·5	35·5	34·5	33·0
128·0	108·0	370·0	300·0	97·0	102·0	71·0?	—	—	—	38·5	—	30·0
127·0	110·0	353·0	286·0	98·0	93·0	62·0	87·5	—	19·0	35·0	35·0	—
133·0	112·0	370·0	—	100·0	92·5	65·5	91·0	—	22·0	37·5	—	32·0
125·0	103·0	350·0	277·0	95·0	94·0	64·0	91·5	120·5	19·5	35·5	—	32·0
136·0	107·0	359·0	297·0	98·0	95·5	68·0	92·0	128·0	23·5	38·5	36·5	31·0
135·0	105·0?	367·0?	302·0	98·0	98·0	67·0	98·5	—	22·0	39·5	37·5	31·5
128·0	122·0	377·5	301·0	104·0	102·5	73·5	101·5	124·0	23·5	37·0	35·5	29·5
119·0	106·0	347·0	280·0	108·5	100·0	70·0	95·0	124·0	21·5	41·0	39·5	33·5
123·0	115·0	367·0	299·0	103·0	96·0	74·0	102·0	132·0	25·5	37·5	36·5	31·5
107·0	129·0	364·0	292·0	103·0	102·0	67·5	95·0	123·0	19·5	38·5	39·5	33·5
138·0	121·0	377·0	—	—	—	—	96·0	—	—	—	—	—
132·0	112·0	367·0	302·0	107·0	106·0	71·0	101·5	—	24·0	37·0	37·5	31·5
134·0	105·0	366·0	297·0	103·0	102·0	68·5	89·5	—	21·5	38·5	35·5	28·5
127·0	125·0	390·0	319·0	97·0	93·5	73·5	94·0	—	23·0	35·5	33·5	33·5
129·0	114·0	374·0	295·0	—	—	66·0	92·5	126·0	21·5	38·5	37·0	29·5
140·0	122·0	392·0	316·0	104·0	105·0	70·0	102·5	129·0	22·5	38·5	36·0	30·0
131·0	109·0	364·0	302·0	102·0	95·0	72·0	89·0	124·0	22·5	38·0	36·5	37·0
140·0	119·0	386·0	288·0	101·0	99·0	68·0	103·0	125·0	21·5	40·5	38·5	32·5
132·0	100·0	357·0	298·0	99·0	95·0	66·5	92·5	123·0	17·5	37·5	37·0	32·5
120·0	110·0	349·0	282·0	95·5	94·0	63·0	95·0	119·0	21·5	35·5	35·0	30·0
124·0	112·0	356·0	301·0	105·0	102·5	64·5	94·0	127·0	20·0	37·5	37·5	33·0
127·0	129·0	377·0	279·0	102·0	105·0	80·0	102·5	126·0	24·5	39·5	37·5	32·0
133·0	119·0	375·0	288·0	98·5	99·0	72·5	87·0	124·0	21·5	40·0	39·5	31·0
135·0	117·0	373·0	283·0	98·5	100·0	63·0	87·0	113·0	20·0	35·5	34·5	30·0
123·0	105·0	350·0	284·0	98·0	98·5	63·5	90·5	—	22·0	36·0	35·0	29·5
131·0	104·0	370·0	301·0	106·5	100·0	70·0	93·5	—	22·0	38·0	38·0	34·5
134·0	111·0	371·5	294·0	92·5	89·5?	57·0?	87·0	—	20·0	—	35·0	29·5
140·0	116·0	380·0	304·0	99·0	94·0	68·5	96·5	124·0	20·0	38·0	37·5	34·5
129·0	112·0	373·0	300·0	100·0	98·0	69·0	93·5	128·0	20·0	38·0	39·5	31·0
135·0	106·0	381·0	310·0	101·0	99·0?	69·0?	87·0	127·0	19·5	39·0	37·5	35·0
147·0	108·0	395·0	306·0	104·0	96·0	63·0	92·5	122·0	22·0	39·5	37·5	28·5
123·0	115·0	357·0	288·5	101·0	93·0	68·5	93·5	120·0	21·5	40·5	40·0	31·5
132·0	99·0	360·0	295·0	99·5	99·5?	63·0?	97·0	134·5	21·0	42·0	40·0	29·5
137·0	111·0	378·0	296·0	105·0	100·5	63·5	90·0	130·0	20·0	41·5	41·0	30·5
123·0	112·0	357·0	286·0	99·0	98·0?	66·0?	92·0	128·0	23·0	36·0	34·5	28·0
121·0	119·0	366·0	288·0	100·0	87·5	72·0	—	—	21·0	—	37·5	—
131·0	129·0	389·0	299·0	105·5	104·0	71·0	103·0	—	23·0	—	—	—
125·0	128·0	380·0	306·0	101·5	102·0	65·5	96·5	118·0	21·5	40·0	40·0	31·0
135·0	114·0	376·0	301·0	105·0	103·0	70·0	105·0	125·0	23·0	40·0	40·5	31·5
130·0	114·0	376·0	300·0	113·0	103·0	74·5	90·0	128·5	25·0	42·0	39·0	32·0
131·5	120·0	392·5	307·0	102·5	106·0	73·5	94·0	136·0	21·0	40·5	40·5	31·5
—	—	—	—	—	—	—	—	—	—	—	—	—
—	—	—	—	101·5	—	69·5?	94·2	115·0	—	—	—	—
126·0	126·0	387·0	—	—	—	—	—	—	—	—	—	—

tter in square brackets denotes that either the measurement is one not usually included in the biometric technique, or else that

APPENDIX I. *Professor D. E. Derry's Measurements of Male Adult Badari Skulls.**

urements						Cranial Indices									
$[O_1'L]$	$[O_2R]$	$[O_2L]$	$[NH, L]$	NB	$[P\angle]$	$100\frac{B}{L}$	$100\frac{H'}{L}$	$100\frac{B}{H'}$	$100\frac{(B-H')}{L}$	$[100\frac{G'H}{GB}]$	$[100\frac{Prosth.\,GL}{LB}]$	$[100\frac{O_2}{O_1'}, R]$	$[100\frac{O_2}{O_1'}, L]$	$[100\frac{NB}{NH}, L]$	$[w_1]$
—	—	—	54·5	25·0	86°·0	70·0	75·6	92·6	−5·6	74·9	—	—	—	45·8	108·5
34·5	33·0	32·0	55·0	28·0	79°·5	69·6	75·0	92·7	−5·5	80·3	100·0	93·0	92·8	50·9	109·0
—	30·0	—	49·0	—	80°·0	74·0	73·7	100·4	+0·3	—	105·1	77·9	—	—	107·5
35·0	—	30·5	43·0	21·0	81°·5	69·0	79·3	87·0	−10·3	70·8	94·9	—	87·4	48·8	101·0
—	32·0	—	45·5	24·5	86°·0	69·0	73·6	93·7	−4·6	72·0	92·5	85·3	—	53·8	99·5
—	32·c	—	46·0	20·5	83°·5	70·3	72·8	96·5	−2·6	69·9	98·9	90·1	—	44·5	101·0
36·5	31·0	31·0	48·5	24·5	82°·5	77·6	71·9	107·9	+5·7	73·9	97·4	80·5	84·9	50·5	111·0?
37·5	31·5	31·0	49·0	23·5	78°·0	71·4	81·8	87·3	−10·4	68·0	100·0	79·7	82·7	47·9	104·0
35·5	29·5	31·0	53·0	26·0	81°·5	74·4	76·3	97·5	−1·9	72·4	98·5	79·7	87·3	49·0	102·0
39·5	33·5	33·5	49·5	29·5	84°·0	67·8	76·1	89·1	−8·3	73·7	92·2	81·7	84·8	59·5	114·5
36·5	31·5	30·0	51·0	25·5	86°·5	75·8	72·8	104·2	+3·0	72·5	93·2	84·0	82·2	50·0	117·5
39·5	33·5	33·5	52·0	23·5	79°·0	78·4	74·3	105·6	+4·1	71·0	99·0	87·0	84·8	45·2	115·5
—	—	—	—	—	—	64·1	—	—	—	—	—	—	—	—	108·5
37·5	31·5	31·0	49·5	25·5	84°·0	75·3	71·5	105·3	+3·8	69·9	99·1	85·1	82·7	51·5	114·0
35·5	28·5	29·0	48·5	23·5	78°·0	72·4	78·4	92·3	−6·1	76·5	99·0	74·0	81·7	48·4	—
33·5	33·5	33·5	51·5	23·0	84°·0	77·3	73·8	104·7	+3·5	78·2	96·4	94·4	100·0	44·7	116·0
37·0	29·5	30·0	46·0	24·0	81°·5	69·9	—	—	—	71·3	—	76·6	81·1	52·2	—
36·0	30·0	31·5	51·5	26·0	81°·5	72·6	73·7	98·6	−1·1	68·3	101·0	77·9	87·5	50·5	109·0
36·5	37·0	36·5	51·5	24·0	85°·0	75·8	75·8	100·0	0·0	80·9	93·1	97·4	100·0	46·6	117·0
38·5	32·5	32·5	47·0	25·0	81°·0	67·2	70·0	95·9	−2·9	66·0	98·0	80·2	84·4	53·2	113·5
37·0	32·5	32·5	49·0	23·5	81°·0	76·0	80·7	94·2	−4·7	71·9	95·9	86·7	87·8	47·9	111·5
35·0	30·0	32·0	44·0	24·0	81°·5	75·7	75·7	100·0	0·0	66·3	98·4	84·5	91·4	54·5	99·0
37·5	33·0	34·0	45·5	24·0	82°·0	77·2	73·0	105·7	+4·2	68·6	97·6	88·0	91·7	52·7	109·0
37·5	32·0	32·0	53·0	25·0	81°·0	66·5	70·2	94·7	−3·7	78·0	102·9	81·0	85·3	47·2	107·0?
39·5	31·0	31·5	49·0	23·5	81°·0	75·3	73·7	102·3	+1·6	83·3	100·5	77·5	79·7	47·9	111·5
34·5	30·0	30·5	43·0	22·5	79°·0	74·6	74·6	100·0	0·0	72·4	101·5	84·5	87·0	52·3	—
35·0	29·5	28·0	45·0	23·0	79°·5	75·4	76·6	98·5	−1·2	70·1	100·5	81·9	80·0	51·1	103·0
38·0	34·5	33·5	47·5	23·5	84°·0	73·0	74·9	97·5	−1·9	74·8	93·8	90·8	88·2	49·5	—
35·0	29·5	30·0	42·0	23·5	—	74·5	77·7	95·9	−3·2	65·5	96·8?	—	85·7	55·9	100·0
37·5	34·5	32·5	48·5	24·5	81°·5	72·6	72·6	100·0	0·0	71·0	94·9	90·8	86·7	50·5	109·5
39·5	31·0	31·0	49·5	25·5	80°·5	73·2	73·4	99·6	−0·3	73·8	98·0	81·6	78·5	51·5	113·5
37·5	35·0	33·5	50·5	27·0	85°·0	74·9	70·1	106·9	+4·9	79·3?	98·0?	89·7	89·3	53·4	108·0
37·5	28·5	28·5	46·5	27·0	81°·5	69·4	77·5?	89·5?	−8·1?	68·1	92·3	72·2	76·0	58·1	116·5
40·0	31·5	31·5	50·5	26·0	85°·5	71·3	73·8	96·6	−2·5	73·2	92·1	77·8	78·7	51·4	109·0?
40·0	29·5	29·5	42·5	25·0	78°·5	73·3	74·1	98·9	−0·8	64·9?	100·0?	70·2	73·7	58·8	117·0
41·0	30·5	30·5	48·0	26·0	82°·0	69·5	72·1	96·3	−2·7	70·5	95·7	73·5	74·4	54·2	118·5
34·5	28·0	28·0	48·0	25·0	82°·0	73·3	72·5	101·2	+0·8	71·7	99·0?	77·8	81·2	52·1	—
37·5	—	34·5	54·0	24·5	86°·5	73·1	71·7	101·9	+1·4	—	87·5	—	92·0	45·4	111·5
—	—	—	51·0	28·0	80°·5	68·9	70·2	98·2	−1·3	68·9	98·6	—	—	54·9	116·0
40·0	31·0	31·0	46·0	25·0	85°·5	76·5	70·6	108·4	+5·9	67·9	100·5	77·5	77·5	54·3	—
40·5	31·5	31·0	52·5	27·0	83°·5	68·7	71·9	95·6	−3·2	66·6	98·1	78·7	76·5	51·4	—
39·0	32·0	35·5	57·0	30·0	89°·0	68·5	69·6	98·5	−1·0	82·8	91·1	76·2	91·0	52·6	—
40·5	31·5	31·5	48·5	24·0	77°·5	74·3	72·2	102·9	+2·1	78·2	103·4	77·8	77·8	49·4	—
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	110·0
—	—	—	46·5	22·5	—	70·6	75·6	93·3	−5·0	73·9?	—	—	—	48·4	—
—	—	—	—	—	—	72·4	75·3?	96·2?	−2·9?	—	—	—	—	—	118·0?

ique, or else that Professor Derry's method of taking the measurement does not accord with biometric practice (see pp. 293—297 of text). Unless otherv

Badari Skulls.*

		Mandibular Measurements								
[100 $\frac{O_2}{O_1}$, L]	[100 $\frac{NB}{NH}$, L]	$[w_1]$	$[w_2]$	h_1	[*Max. B ramus*]	rb'	$[M\angle]$	$[rl]$	$[c_p l]$	
—	45·8	108·5	87·0	31·5	42·5	31·0	118°·0	63·0	76·0	Facial skeleton somewhat distorted posthumously. Maxillae
92·8	50·9	109·0	71·0	32·5	45·0	34·0	110°·0	72·0	84·0	Maxillae—$m_1 r$, $m_2 r$, $m_3 r$, $m_3 l$. The loss of the upper molars
—	—	107·5	90·0	33·5	39·5	33·0	125°·5	58·5	77·0	Maxillae—$m_3 r$, left side missing; mandible—$m_1 r$, $m_2 r$, $m_3 r$,
87·4	48·8	101·0	96·0	30·5	41·5	34·0	117°·0	54·0	73·0	Mandible—$m_1 r$. Teeth very worn. Beard.
—	53·8	99·5	82·5	32·0	43·0	32·0	116°·5	55·0	77·5	Teeth very worn.
—	44·5	101·0	89·0	32·5	38·0	33·5	116°·5	57·0	80·0	Teeth little worn.
84·9	50·5	111·0?	93·0	29·5	—	—	105°·0	69·0	86·5	Maxillae—$m_2 l$, $m_3 l$. Teeth very worn.
82·7	47·9	104·0	93·5	34·0	44·0	36·5	113°·0	60·0	84·0	Epipteric bone on left side. Teeth little worn.
87·3	49·0	102·0	92·0	37·5	46·0	35·0	135°·0	58·0	81·5	Teeth moderately worn. Negroid?
84·8	59·5	114·5	89·5	33·5	43·0	32·0	123°·0	62·0	76·5	Maxillae—$m_1 r$, $m_1 l$, $m_2 l$, $m_3 l$; mandible—$i_2 r$, cr. Teeth wor
82·2	50·0	117·5	96·0	33·5	42·0	36·0	110°·5	64·0	85·0	Teeth very worn. Aged.
84·8	45·2	115·5	84·5	27·5	45·0	35·0	110°·0	66·0	84·0	Mandible—$m_1 r$, $m_2 r$; no third molars in either jaw. Teeth v
—	—	108·5	84·0	31·0	43·5	37·0	116°·5	65·5	81·5	Maxillae—no third molars. Teeth very worn.
82·7	51·5	114·0	90·5	35·5	47·5	39·0	118°·0	60·5	86·0	Mandible—$i_2 r$ not present, cr not erupted but visible throug
81·7	48·4	—	—	—	—	—	—	—	—	No mandible. Teeth well worn.
100·0	44·7	116·0	96·5	33·5	39·0	33·0	124°·0	60·0	78·0	Teeth little worn.
81·1	52·2	—	—	—	—	—	118°·0	55·0	79·0	Many teeth lost and those remaining worn to stumps.
87·5	50·5	109·0	81·0	34·0	42·0	37·0	115°·5	68·0	84·0	Teeth much worn.
100·0	46·6	117·0	96·5	32·0	42·5	35·0	123°·0	56·5	73·0	Mandible—$m_3 l$ not erupted and no space for it. Teeth well
84·4	53·2	113·5	97·5	27·5	45·5	37·0	114°·5	70·0	80·0	Maxillae—$m_1 r$, $m_2 r$, $m_2 r$, $m_1 l$, $m_2 l$, $m_3 l$; mandible—several t
87·8	47·9	111·5	87·0	27·5?	40·0	32·0	112°·5	62·0	78·0	Teeth very worn.
91·4	54·5	99·0	85·0	29·5	37·5	31·5	128°·0	49·0	72·0	No third molars in either jaw. Remains of straight hair tu
91·7	52·7	109·0	92·0	28·0	42·0	37·5	114°·5	61·0	76·0	Teeth very worn.
85·3	47·2	107·0?	88·5	38·0	43·5	35·5	119°·5?	62·0?	85·0	Teeth much worn. Negro?
79·7	47·9	111·5	97·5	36·0	45·0	34·5	122°·0	59·0	76·0	Maxillae—$pm_2 l$, $m_1 l$. Teeth very worn.
87·0	52·3	—	—	—	—	—	—	—	—	No mandible. Maxillae—$m_3 r$. Teeth well worn.
80·0	51·1	103·0	87·0	31·5	38·0	32·0	119°·5	58·5	77·0	Metopic. Maxillae—no third molars; mandible—$m_1 r$, no rig
88·2	49·5	—	—	—	—	—	—	—	—	No mandible. Teeth moderately worn.
85·7	55·9	100·0	82·0	—	37·0	32·5	116°·5	62·0	76·0	Maxillae—all except a few front teeth lost; mandible—all t
86·7	50·5	109·5	96·5	31·0	41·5	29·0	119°·5	62·5	78·5	Mandible—$m_1 r$. Teeth very worn. Male hip bone.
78·5	51·5	113·5	94·0?	33·0	46·0	37·5	113°·0	60·5	82·0	Maxillae—absence of $i_2 r$ and no space for it, $i_2 l$ is a small p
89·3	53·4	108·0	94·0	—	43·0	33·5	129°·5	—	75·0	Maxillae—all teeth lost from right side and a few stumps on mandibular joints. Aged.
76·0	58·1	116·5	92·0	30·0	45·5	37·5	104°·0	62·0	84·0	Some posthumous distortion. Hair curly and in waves but
78·7	51·4	109·0?	91·5	33·5	35·5	31·0	113°·5	62·5	81·0	Most teeth lost and a few stumps remaining.
73·7	58·8	117·0	96·5	30·5	44·0	32·0	117°·0	57·0	82·5	A few stumps only remaining. Enormous superciliary ridg
74·4	54·2	118·5	97·5	30·5	44·0	36·0	116°·0	65·0	78·5	Maxillae—many teeth lost and the lateral incisors appear
81·2	52·1	—	87·5	—	41·5	32·0	123°·0	55·0	72·0	Maxillae—all lost except a few stumps in front; mandible—
92·0	45·4	111·5	97·5	33·0	35·0	30·0	114°·5	61·0	79·0	Maxillae—all molars lost and front teeth worn to stumps.
—	54·9	116·0	101·0	36·5	49·5	38·5	115°·0	69·0	91·0	Mandible—$pm_2 r$, $m_1 r$ and $pm_2 l$ only *left* as carious stumps.
77·5	54·3	—	—	30·5	43·0	31·0	125°·5?	—	—	Maxillae—all molars lost except $m_1 r$; mandible—only right
76·5	51·4	—	—	—	—	—	—	—	—	No mandible, the mandible with the same number as the s
91·0	52·6	—	89·0	35·0	45·5	37·0	108°·5	65·0	87·0	Maxillae—$i_2 r$. Teeth very worn and front teeth irregular.
77·8	49·4	—	94·5	40·5	—	38·0	—	—	—	Rami of mandible, of which the upper parts are defective,
—	—	110·0	99·0	34·5	48·5	36·0	125°·0	59·0	80·0	Teeth very worn and sockets of some exhibit marked perio
—	48·4	—	100·5	32·0	38·0	31·5	118°·0?	—	77·0	Facial skeleton broken away. Teeth very worn. Male hip b
—	—	118·0?	102·5	34·0	41·0	33·5	123°·5	55·5	82·0	Mandible—all molars lost on left. Teeth worn to stumps.

—297 of text). Unless otherwise stated the mandible is present and no teeth had been lost from either jaw before death. The teeth noted are those lost before

Remarks*
mously. Maxillae—no third molars; mandible—$m_3 l$. Young adult.
he upper molars may have been due to some injury of the jaw.
—$m_1 r$, $m_3 r$, $m_2 r$, cl. Teeth very worn.
rn.
$_2 r$, cr. Teeth worn to stumps. Aged.
her jaw. Teeth very worn.
n.
ut visible through break in bone, $pm_1 r$ rotated, cl represented by small stump possibly the deciduous canine.
to stumps.
or it. Teeth well worn.
ndible—several teeth lost. Teeth very worn.
straight hair turning grey.
vorn.
dible—$m_1 r$, no right third molar, $m_2 l$ present.
; mandible—all teeth lost. Aged. Male hip bone.
p bone.
it, $i_2 l$ is a small peg-shaped tooth. Teeth very worn. Male hip bone.
l a few stumps only remain on left. Mandible—a few stumps on right and all lost on left. Arthritic temporo-
nd in waves but not negroid. Male hip bone.
ng.
superciliary ridges and orbits low and elliptical (Australoid). Male hip bone.
incisors appear never to have been present. Male hip bone.
front; mandible—most teeth not lost. Male hip bone. Grey hair. Aged.
worn to stumps. Male hip bone.
s carious stumps. Very prominent superciliary ridges. Aged.
ndible—only right side present. Teeth very worn. Aged.
number as the skull not belonging to it. Maxillae—all lost except a few stumps. Aged.
teeth irregular. Aged.
rts are defective, are broad and vertical as is characteristic of the Tasian specimens.
ibit marked periostitis.
worn. Male hip bone. Aged.
vorn to stumps.

those lost before death.

To face p. 308.

UNIV. OF MICH.

Locality	Grave No.	Calvarial Measurements								
		L	*B*	*H′*	[*OH*]	*B′*	*U*	S_1	S_2	S_3
Female										
Mostagedda	28/302	175·5	123·0	131·0	112·0	86·0	480·0	120·0	136·0	118·0
,,	28/304	170·0	126·5	136·0	116·5	87·5	483·5	121·0	134·0	111·0
,,	28/307	170·5	127·0	126·0	109·0	80·0	479·0	124·0	118·0	116·0
,,	28/428	171·0	127·0	126·5	110·0	91·0	479·0	121·0	123·0	106·5
,,	28/457	173·0	123·5	127·5	108·0	92·0	480·0	124·0	116·0	104·0
,,	28/460	172·0	120·5	129·0	107·0	91·0	476·0	119·0	122·0	105·0
,,	28/470	169·0	127·5	123·0	104·0	88·5	471·0	115·0	109·0	114·5
,,	28/475	171·0	134·0	132·5	114·0	82·0	489·0	122·0	117·5	123·0
,,	28/480	174·0	130·0	129·0	112·5	90·0	483·0	—	—	—
,,	28/483	173·0	131·5	126·5	112·5	93·0	489·0	120·0	128·0	109·0
,,	28/1200	177·0	120·5?	133·5?	114·5	89·5	485·0	117·0	135·0	116·0
,,	28/1206 A	174·0	132·5	120·0	107·0	94·5	500·0	129·0	120·0	115·0
,,	28/1212	177·5	129·0	133·0	115·5	96·0	499·0	123·0	135·0	111·0
,,	28/1214	177·5	124·0	132·0	113·0	90·5	488·0	117·0	130·0	114·0
,,	28/1242	176·5	127·0	127·5	112·5	87·5	489·0	120·0	132·0	116·0
,,	28/2001	180·5	120·0	133·0	115·0	90·5	491·0	122·0	133·0	105·0
,,	28/2242	173·0	128·0	129·0	111·0	91·0	490·0	119·0	128·0	108·0
,,	28/2677	172·0	129·0	131·0	111·0	88·0	481·5	116·0	107·0	124·0
,,	29/2700	175·0	122·5	127·5	108·0	88·0	480·0	124·0	124·0	112·0
,,	29/3202	—	—	—	—	84·0	—	—	—	—
,,	29/3502	180·0	130·5	138·0	116·5	92·0	495·0	129·0	135·0	117·0
,,	29/3508	173·5	131·0	129·0	110·0	88·0	490·0	118·0	136·0	114·0
,,	29/3527 B	177·0	118·5	134·5	114·0	86·5	483·0	122·0	123·0	115·0
,,	29/3530	172·0	128·0	124·0	109·0	85·5	478·5	124·0	126·0	102·0
,,	29/3533	169·0	124·5	133·5	112·0	85·0	472·0	127·5	119·0	107·0
,,	29/3536	175·0	130·0	129·0	112·0	88·0	485·0	124·0	122·0	110·0
,,	29/3538	173·0	127·5	134·0	115·0	93·0	490·0	121·0	129·0	119·0
,,	29/3550	172·5	127·0	128·5	110·0	83·0	477·0	125·0	123·0	113·0
,,	29/3710	175·0	124·5	137·0	115·5	85·0	474·0	125·0	126·0	106·0
,,	29/10013	176·5	132·5	127·0	111·0	99·0	495·0	130·0	124·0	111·0
Badari	30/2015	180·0	133·0	130·0	114·0	91·5	500·0	126·0	128·0	112·0
,,	30/2030	175·0	137·5	125·0	112·0	87·5	492·0	117·0	127·0	112·0
,,	30/2032	178·5	136·0	130·0	111·5	90·0	496·0	128·0	115·0	120·0
,,	30/2500	176·0	127·5	134·0	113·0	88·0	484·0	124·0	125·0	109·0
Juvenile										
Mostagedda (♂)	29/3527 A	174·0	132·0	131·5	114·0	84·5	481·0	123·0	132·0	106·0
,, (♀)	29/3510	177·0	123·0	123·0	112·0	87·0	485·0	130·0	123·0	112·5
,, (,,)	28/493	170·0	135·5	127·0	110·0	83·5	480·0	119·0	120·0	113·0

* An index letter in square brackets

Measurements							F				
	S_1	S_2	S_3	S	[*Broca's Q'*]	*LB*	[*Pros-thion GL*]	[*G'H*]	*GB*	*J*	[*DC*]
	120·0	136·0	118·0	374·0	281	93·5	88·0	60·0	84·0	—	19·0
	121·0	134·0	111·0	366·0	292	96·0	96·0	65·0	87·0	116·0	19·0
	124·0	118·0	116·0	358·0	282	94·0	92·5	64·0	87·5	119·5	19·5
	121·0	123·0	106·5	350·5	282	93·0	98·5	59·5	95·0	—	22·0
	124·0	116·0	104·0	344·0	272	101·0	98·5	63·0	92·5	118·0	21·0
	119·0	122·0	105·0	346·0	271	100·5	97·0	69·5	91·0	108·5	21·5
	115·0	109·0	114·5	338·5	273	93·5	88·5	66·0	93·0	119·0	23·0
	122·0	117·5	123·0	362·5	301	94·0	93·0	67·0	—	—	21·5
	—	—	—	—	—	97·5	94·0	62·0	90·0	—	20·5
	120·0	128·0	109·0	357·0	290	95·0	94·0	66·0	94·5	116·0	24·0
	117·0	135·0	116·0	368·0	292	97·0	98·0	66·0	94·5	—	20·5
	129·0	120·0	115·0	364·0	287	89·0	85·0	62·0	84·0	120·5	21·0
	123·0	135·0	111·0	369·0	285	97·0	95·0	70·5	90·0	118·0	23·0
	117·0	130·0	114·0	361·0	281	100·5	97·0	63·5?	92·0	—	—
	120·0	132·0	116·0	368·0	286	92·5	89·0	64·5	89·5	113·0	22·5
	122·0	133·0	105·0	360·0	285	100·0	97·0	67·0	90·5	—	—
	119·0	128·0	108·0	355·0	287	99·5	99·0	66·0	97·0	122·0	22·5
	116·0	107·0	124·0	347·0	279	98·0	97·0	66·0	99·0	122·5	17·5
	124·0	124·0	112·0	360·0	276	93·0	90·5	61·0	90·5	122·5	19·0
	—	—	—	—	—	—	—	64·5	102·5	—	20·0
	129·0	135·0	117·0	381·0	291	100·5	97·5	68·0	101·5	118·5	25·0
	118·0	136·0	114·0	368·0	290	93·0	90·0	60·5	87·5	118·5	20·5
	122·0	123·0	115·0	360·0	282	98·0	96·0	66·5	97·5	114·0	21·5
	124·0	126·0	102·0	352·0	276	94·0	98·0	67·0	86·5	120·0	23·5
	127·5	119·0	107·0	353·5	278	97·0	95·5	67·5	89·0	114·0	21·0
	124·0	122·0	110·0	356·0	275	95·5	93·5	65·0	87·0	113·5	21·0
	121·0	129·0	119·0	369·0	275	92·5	92·0	65·0	98·0	121·0	19·0
	125·0	123·0	113·0	361·0	263	94·0	93·5	61·0	86·5	107·0	19·5
	125·0	126·0	106·0	357·0	271	99·5	90·5	63·0	92·0	116·0	21·0
	130·0	124·0	111·0	365·0	287	96·5	93·0	65·5	96·0	123·5	23·5
	126·0	128·0	112·0	366·0	299	101·5	100·0	67·5	96·0	119·0	20·0
	117·0	127·0	112·0	356·0	292	95·0	87·5	68·0	97·5	123·0	19·5
	128·0	115·0	120·0	363·0	290	100·0	93·0	68·0	96·5	121·5	20·5
	124·0	125·0	109·0	358·0	284	102·0	102·0	70·0	94·0	124·0	21·5
	123·0	132·0	106·0	361·0	292	99·0	98·0	74·0	89·0	112·0	23·0
	130·0	123·0	112·5	365·5	280	93·0	101·5	64·0	87·5	109·0	19·0
	119·0	120·0	113·0	352·0	280	95·0	91·0	60·5	86·0	110·0	18·0

n index letter in square brackets denotes that either the measurement is one not usually included in the biom

APPENDIX II. *Professor D. E. Derry's Measurements of Female Adult and Ju*

Facial Measurements									Cranial Indices						
J	[*DC*]	[$O_1'R$]	[$O_1'L$]	[O_1R]	[O_1L]	[*NH*, *L*]	*NB*	[*P*∠]	100 $\frac{B}{L}$	100 $\frac{H'}{L}$	100 $\frac{B}{H'}$	100 $\frac{(B-H')}{L}$	[100 $\frac{G'H}{GB}$]	[100 $\frac{Prosth.\ GL}{LB}$]	[100 $\frac{O_2}{O_1'}$, *R*]
—	19·0	39·0	39·0	32·5	33·5	44·5	23·0	84°·5	70·1	74·6	93·9	−4·6	71·4	94·1	83·3
116·0	19·0	38·5	37·0	34·0	34·0	48·0	22·5	77°·5	74·4	80·0	93·0	−5·6	74·7	100·0	88·3
119·5	19·5	36·0	36·0	31·5	32·0	48·5	24·0	81°·0	74·5	73·9	100·8	+0·6	73·1	98·4	87·5
—	22·0	37·0	37·0	31·5	30·5	44·0	25·0	75°·0	74·3	74·0	100·4	+0·3	62·6	105·9	85·1
118·0	21·0	37·5	37·0	32·0	30·0	46·5	26·0	80°·5	71·4	73·7	96·9	−2·3	68·1	97·5	85·3
108·5	21·5	39·0	38·0	34·0	34·0	51·0	28·5	79°·5	70·1	75·0	93·4	−4·9	76·3	96·5	87·2
119·0	23·0	39·0	38·0	32·5	31·5	47·0	24·5	82°·0	75·4	72·8	103·7	+2·6	70·9	94·6	83·3
—	21·5	—	33·0	—	29·0	45·5	25·0	82°·5	78·4	77·5	101·1	+0·9	—	98·9	—
—	20·5	37·5	36·5	30·0	30·0	45·5	24·0	85°·5	74·7	74·1	100·8	+0·6	68·9	96·4	80·0
116·0	24·0	36·0	34·5	30·0	30·5	46·0	24·0	85°·5	76·0	73·1	104·0	+2·9	68·9	98·9	83·3
—	20·5	38·5	37·0	36·0	36·0	47·0	25·5	76°·5	68·1?	75·4?	90·3?	−7·3?	69·8	101·0	93·5
120·5	21·0	37·0	35·5	33·5	33·0	47·5	23·0	86°·0	76·1	68·9	110·4	+7·2	73·8	95·6	90·5
118·0	23·0	38·0	35·5	33·0	32·5	50·0	28·0	83°·0	72·7	74·9	97·0	−2·3	78·3	97·9	86·8
—	—	36·5	35·5	33·5	34·5	45·5	24·5	78°·0	69·8	74·3	93·9	−4·5	69·0?	96·5	91·8
113·0	22·5	35·0	35·0	31·5	31·5	44·5	24·5	83°·0	71·9	72·2	99·6	−0·3	72·1	96·2	90·0
—	—	—	—	—	—	48·5	26·0	80°·5?	66·5	73·7	90·2	−7·2	74·0	97·0	—
122·0	22·5	39·5	39·5	35·0	34·0	46·0	27·0	77°·0	73·9	74·5	99·2	−0·6	68·0	99·5	88·6
122·5	17·5	36·5	36·5	32·5	32·5	50·0	27·0	78°·5	75·0	76·2	98·5	−1·2	66·6	99·0	89·0
122·5	19·0	35·5	35·0	30·0	30·0	44·5	23·0	80°·0	70·0	72·8	96·1	−2·9	67·4	97·3	84·5
—	20·0	34·5	32·0	28·0	28·0	43·0	24·5	—	—	—	—	—	62·9	—	81·2
118·5	25·0	38·0	37·0	33·0	33·0	48·5	24·5	79°·5	72·5	76·6	94·6	−4·2	67·0	97·0	86·8
118·5	20·5	35·0	35·0	30·5	30·5	44·5	23·5	81°·0	75·5	74·3	101·6	+1·2	69·1	96·8	87·1
114·0	21·5	34·0	34·0	32·0	30·0	47·5	22·5	77°·5	66·9	75·9	88·1	−9·0	68·2	97·9	94·1
120·0	23·5	39·0	39·0	33·0	32·5	46·0	22·0	77°·0	74·4	72·1	103·2	+2·3	77·4	104·2	84·6
114·0	21·0	38·0	37·0	34·0	34·0	45·0	25·0	77°·0	73·6	78·9	93·3	−5·3	75·8	98·4	89·5
113·5	21·0	37·0	36·0	32·5	32·5	47·0	23·5	80°·5	74·3	73·7	100·8	+0·6	74·7	97·9	87·8
121·0	19·0	39·0	39·0	31·0	30·0	47·0	22·5	77°·0	73·7	77·4	95·1	−3·8	66·3	99·4	79·5
107·0	19·5	34·0	34·0	30·5	30·0	43·0	21·0	79°·5	73·6	74·5	98·8	−0·9	70·5	99·5	89·7
116·0	21·0	36·0	35·5	32·5	32·0	47·0	23·5	84°·0	71·1	78·2	90·9	−7·1	68·4	90·9	90·3
123·5	23·5	39·5	38·5	32·5	32·5	47·5	24·0	84°·5	75·0	71·9	104·3	+3·1	68·2	96·4	82·3
119·0	20·0	39·5	—	33·5	33·0	48·0	24·0	84°·5	73·9	72·2	102·3	+1·7	70·3	98·5	84·8
123·0	19·5	38·0	38·0	35·5	35·5	50·0	23·5	88°·0	78·6	71·4	110·0	+7·1	69·7	92·1	93·4
121·5	20·5	36·0	36·5	28·5	29·5	50·0	23·5	90°·0	76·2°	72·8	104·6	+3·4	70·4	93·0	79·2
124·0	21·5	36·0	35·0	29·5	30·5	46·0	23·0	82°·0	72·4	76·1	95·1	−3·7	74·4	100·0	81·9
112·0	23·0	36·0	34·5	31·5	32·5	51·5	23·0	83°·0	75·8	75·5	100·4	+0·3	83·1	99·0	87·5
109·0	19·0	—	—	—	—	46·5	23·0	76°·5	69·5	69·5	100·0	0·0	73·1	109·1	—
110·0	18·0	33·5	34·0	30·0	30·5	44·0	23·5	84°·0	79·7	74·7	106·7	+5·0	70·3	95·8	89·6

uded in the biometric technique, or else that Professor Derry's method of taking the measurement does not accord with biometric practice (see pp. 293—

*emale Adult and Juvenile Badari Skulls.**

al Indices					Mandibular Measurements								
[100 $\frac{FH}{GB}$]	[100 $\frac{Prosth.\ GL.}{LB}$]	[100 $\frac{O_2}{O_1'}$, R]	[100 $\frac{O_2}{O_1'}$, L]	[100 $\frac{NB}{NH}$, L]	[w_1]	[w_2]	h_1	[*Max. B ramus*]	rb'	[$M\angle$]	[rl]	[$c_2 l$]	
71·4	94·1	83·3	85·9	51·7	97·0	84·0	—	40·5	36·5	114°·0	52·0	73·5	Most teeth lost and those remaining wo
74·7	100·0	88·3	91·9	46·8	109·5	81·5	27·0	43·0	32·0	110°·5	62·0	80·5	Third molars suppressed in both jaws e
73·1	98·4	87·5	88·9	49·5	107·0	87·0	28·5?	41·0	32·0	108°·0	54·0	77·5	Teeth worn to stumps and several absc
62·6	105·9	85·1	82·4	56·8	103·5	79·5	30·0	44·0	37·5	112°·0	54·0	82·0	Teeth little worn.
68·1	97·5	85·3	81·1	55·9	102·0	86·0	26·5	41·0	34·5	115°·0	57·0	72·0	Teeth much worn and abscess cavity at
76·3	96·5	87·2	89·5	55·9	101·0?	86·5	30·0	38·5	30·5	129°·5	56·0	72·5	Teeth very worn. Maxillae—$i_2 l$ absent
70·9	94·6	83·3	82·9	52·1	—	—	—	—	—	—	—	—	No mandible. Maxillae—$pm_2 l$, $pm_2 r$ an
—	98·9	—	87·9	54·9	—	83·0	32·5	38·5	29·0	125°·0	53·5	71·0	Maxillae—two molars lost; mandible—
68·9	96·4	80·0	82·2	52·7	108·0?	88·5	28·5	40·0*r*	29·0*r*	123°·0	49·5	66·0	Abscess cavity at root of lower $m_1 l$.
68·9	98·9	83·3	88·4	52·2	101·0	81·0	31·5	41·0	33·0	128°·5	50·0	75·5	Maxillae—$m_2 r$; mandible—$m_1 r$, $m_1 l$, m_2
69·8	101·0	93·5	97·3	54·2	104·5	90·5	32·5	39·0	29·0	132°·0	55·0	71·0	Somewhat distorted by grave pressure.
73·8	95·6	90·5	93·0	48·4	104·0	85·0	28·0	38·0	32·5	120°·0	50·0	72·0	Mandible—deciduous second molars pr
78·3	97·9	86·8	91·5	56·0	107·0	88·0	33·5	35·0	29·5	123°·5	54·0	76·0	Mandible—$m_1 l$. Teeth well worn. Meto
69·0?	96·5	91·8	97·2	53·8	100·0	87·5	—	38·0	33·0	129°·0	52·5	73·0	Slightly sprung. Young adult.
72·1	96·2	90·0	90·0	55·0	101·5	80·0	28·5	37·5	30·0	122°·0	48·0	70·0	Teeth worn to stumps and most lost be
74·0	97·0	—	—	53·6	97·0	93·5	30·5	42·0	32·0	123°·5	62·0	74·5	Teeth worn to stumps and several lost
68·0	99·5	88·6	86·1	58·7	—	—	31·0	38·5	29·0	132°·0*r*	56·5	77·5	Teeth worn to stumps and several lost
66·6	99·0	89·0	89·0	54·0	114·0	87·5	—	40·0	34·5	113°·0	59·0	83·5	Teeth much worn. Believed pregnant
67·4	97·3	84·5	85·7	51·7	104·0	95·5	30·5	41·0	30·5	126°·5	50·0	70·0	All molars lost before death except low
62·9	—	81·2	87·5	57·0	95·0?	88·0	33·5	40·5	32·0	125°·0	55·0	74·0	Skull in pieces; only face and frontal b
67·0	97·0	86·8	89·2	50·5	100·0	87·5	31·0	38·0	32·0	116°·0	61·5	77·0	Mandible—$m_2 l$, $m_2 l$; abscess cavities at
69·1	96·8	87·1	87·1	52·8	110·0	89·5	31·5	41·5	36·0	122°·5	54·0	74·0	Teeth little worn.
68·2	97·9	94·1	88·2	47·4	103·5	76·0	37·0	42·5	32·5	122°·5	53·0	77·0	Maxillae—$m_1 r$, $i_1 l$; mandible—front tee
77·4	104·2	84·6	83·3	47·8	106·0	84·5	32·5	40·5	33·5	122°·5	52·0	77·5	Hair wavy. Female hip bone.
75·8	98·4	89·5	91·9	55·5	98·0	79·5	34·0	40·0	28·5	137°·0	52·0	75·0	Maxillae—$m_1 r$, $m_1 l$, $m_2 l$, $m_2 l$; mandibl
74·7	97·9	87·8	90·3	50·0	108·5	86·5	27·5	42·0	29·5	128°·0	56·0	69·5	Maxillae—$m_1 l$, $m_2 l$, $m_2 l$; mandible—m_1
66·3	99·4	79·5	76·9	47·8	97·0	78·0	30·0	40·5	36·0	116°·5	58·0	81·0	Young adult. Female hip bone.
70·5	99·5	89·7	88·2	48·8	92·0	76·5	28·0	39·0	30·0	118°·0	52·0	73·0	Teeth very worn and several abscess ca
68·4	90·9	90·3	90·1	50·0	107·0	90·0	—	39·5	32·5	120°·5	55·5	76·0	Maxillae—all molars; mandible—all ex
68·2	96·4	82·3	84·4	50·5	105·5	88·0	35·0	41·5	34·0	115°·0	57·5	81·5	Teeth well worn. Both pterygoid lami
70·3	98·5	84·8	—	50·0	110·5	85·0	31·0	41·0	35·0	107°·0	60·0	81·0	Maxillae—$m_2 l$, $m_2 l$. Teeth worn to stu
69·7	92·1	93·4	93·4	47·0	—	—	28·5	39·0	31·0	115°·0	58·0	76·5	Maxillae—$m_2 l$, $m_2 l$. Right side of man
70·4	93·0	79·2	80·8	47·0	112·0	92·0	32·0	45·0	37·0	119°·5	58·0	75·0	Maxillae—$m_2 r$, $m_2 r$. Teeth well worn a
74·4	100·0	81·9	87·1	50·0	116·0	93·5	33·0	40·0	32·5	120°·0	58·0	79·0	Teeth well worn. Female hip bone.
83·1	99·0	87·5	94·2	44·7	96·0	88·0	35·5	36·5	29·0	128°·5	55·5	73·0	All teeth erupted but age supposed 15 pubis with ischium and ilium still
73·1	109·1	—	—	49·5	97·0	80·5	29·0	39·0	35·5	110°·5	59·0	78·0	Young adult dwarf. Right epiphysis o somewhat resemble those met wit
70·3	95·8	89·6	89·7	53·4	91·0	72·0	30·0	36·0	31·0	121°·5	48·0	72·0	Third molars not fully erupted.

etric practice (see pp. 293—297 of text). Unless otherwise stated the mandible is present and no teeth had been lost from either jaw before death. The teeth noted a

Remarks*
ose remaining worn to stumps. Sutures almost obliterated but metopic visible. Aged.
ed in both jaws except lower $m_3 r$.
and several abscess cavities. Aged.
abscess cavity at root of upper $m_1 l$.
cillae—$i_2 l$ absent and its site occupied by c; mandible—front teeth crowded on left.
e—$pm_2 l$, $pm_3 r$ and $m_3 r$ absent and possibly never erupted.
lost; mandible—$m_1 r$, $m_1 l$, $m_2 l$. Teeth very worn.
of lower $m_1 l$.
ble—$m_1 r$, $m_1 l$, $m_2 l$. Teeth worn to stumps.
y grave pressure.
second molars present. Metopic. Epipteric bone on each side.
well worn. Metopic. Fronto-temporal articulation on both sides.
ng adult.
and most lost before death. Both tympanic plates perforated.
and several lost before death. Upper $i_1 r$ not erupted. Somewhat distorted posthumously. Aged.
and several lost before death. Aged.
lieved pregnant at time of death; bones of foetus in grave.
death except lower $m_3 l$. Female hip bone.
ace and frontal bone intact. Female hip bone.
bscess cavities at site and at root of cl. Teeth very worn. Female hip bone.
ndible—front teeth crowded and only three incisors. Abscess cavities. Fronto-temporal articulation on both sides. Female hip bone.
ip bone.
$_2 l$, $m_3 l$; mandible—$m_1 r$, $m_1 l$, $m_2 l$. Female hip bone.
$_3 l$; mandible—$m_1 l$, $m_2 l$, $m_3 l$. Teeth very worn. Female hip bone.
hip bone.
several abscess cavities. Female hip bone.
mandible—all except molars. Thinning and depression of parietals. Aged. Female hip bone.
h pterygoid laminae and right jugular process perforated. Cribra orbitalia on left. Hair wavy. Female hip bone.
eeth worn to stumps. Depressed and comminuted fracture on frontal bone. Old. Female hip bone.
ight side of mandible missing. Teeth well worn. Female hip bone.
Teeth well worn and crowded in front. Female hip bone.
nale hip bone.
age supposed 15 or 16. Basilar suture widely open and crest of ilium and ischio-pubic epiphyses not united. Line of union of m and ilium still visible. Hair curly but not negroid.
Right epiphysis of acromion of scapula not united. Epiphysis of clavicle not completely united. Proportions of upper limb bones ble those met with in achondroplasia.
y erupted.

The teeth noted are those lost before death.

To face p. 308.

type in being less prognathous, and in having lower nasal but higher cephalic indices.

(ii) Close connections can be found between these later Egyptian types and some representing European populations in contemporary or later times, but the Badari and other Predynastic Egyptian types bear no close resemblance to any recorded for a European population other than that of modern Sardinians. The connection between the Badari and Sardinian series is not very close, but it is closer than that found between the Predynastic and some late Dynastic Egyptian series.

(iii) Connections of the same order are found between the Badari and all the modern Indian series available, the most intimate being with a Dravidian series. The other Predynastic Egyptian types are rather further removed from the Indian, and between the last and the Dynastic Egyptians no close resemblances are found.

(iv) A wider gap separates all the Egyptian from African negro types, but the resemblances between these and the Badari and other Predynastic types are appreciably closer than those between the negro and late Dynastic Egyptian types.

The interpretation of the observed situation in terms of racial descent must necessarily be hazardous in the present state of our knowledge, and far more material than that available at present may be needed in order to reveal the actual relationships, which are almost certainly complex, between the populations represented. Meanwhile it is of interest to note that the Badari racial type, which is believed to be the earliest of which we have any adequate knowledge of a statistical kind, is also peculiarly generalised.

I.

THE RATIO OF THE MEAN DEVIATION TO THE STANDARD DEVIATION AS A TEST OF NORMALITY.

By R. C. GEARY, M.Sc.

THE problem to be considered in this paper is the following. A sample of n' elements with measures $x_1, x_2, \ldots x_{n'}$, is drawn at random from a universe which is presumed continuous: to determine from the sample whether the universe may be regarded as normal, that is to say whether the probability of drawing therefrom an element, measuring from $x - \frac{\delta x}{2}$ to $x + \frac{\delta x}{2}$, when δx is small, is

$$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\alpha)^2}{2\sigma^2}} \delta x \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Two cases are considered: (i) that in which the universal mean α may be presumed known, but the standard deviation σ is unknown, and (ii) that in which both the mean and standard deviation are unknown.

The usual method of dealing with this problem, and that which in principle is adopted here, is that of devising a function, or functions, of the sample variables $x_1, x_2, \ldots x_{n'}$, which assume characteristic values for infinite samples (i.e. $n' = \infty$) drawn from a normal universe. We try to find the actual or approximate frequency distributions, for different sizes of sample, on the assumption that the unknown parent universe is normal. If, for the actual sample, the values of such functions are very improbable, we decide that the universe from which the sample was drawn is not normal.

It is quite obvious that in problems of this kind we can only determine "necessary" (and not "sufficient") conditions. We may hope to be in a position to say that the parent universe was *not* A; we can never say on the evidence of the limited sample alone that it probably *was* drawn from a universe B. In the first place, the number of *independent* tests which may be applied is strictly limited: it cannot exceed n', the number in the sample, when the parent universe is completely known. In fact, if n' algebraically independent functions of the n' variables be constructed, all other functions of the variables will be algebraically related to these n' functions, and their frequency distributions will in consequence be absolutely determined by the multiple frequency distribution of the n' functions. If all the n' tests showed that the given sample could be regarded as not improbably having been drawn from a universe B, it could equally plausibly be

regarded as having been drawn from an infinity of other universes differing from B (in functional form, or in the values of the parameters) to a greater or lesser extent according as n' is small or large.

The range of choice of possible tests of normality is extensive. Clearly it is desirable that the test functions should satisfy the following conditions:

(i) They should be as simple as possible, not only for convenience of calculation but because the simpler they are the more likely is it that their frequency distributions, or at least their moments and Thiele semi-invariants, can be determined.

(ii) They should be symmetrical in the sampled variables so as to avoid bias.

(iii) When n' tends towards infinity they should assume values characteristic of the normal universe.

(iv) They should be independent of the unknown parameters α and σ [see (1) above], in the sense of being invariant for the transformation

$$x_i' = \frac{x_i - \alpha}{\sigma} \quad (i = 1, 2, \ldots, n').$$

Hitherto* the functions $\sqrt{\beta_1}$ and β_2, with

$$\beta_1 = m_3^2/m_2^3, \quad \beta_2 = m_4/m_2^2,$$

and $n'm_2 = \sum_{i=1}^{n'} (x_i - \bar{x})^2$, $n'm_3 = \sum_{i=1}^{n'} (x_i - \bar{x})^3$, $n'm_4 = \sum_{i=1}^{n'} (x_i - \bar{x})^4$, $n'\bar{x} = \sum_{i=1}^{n} x_i$,

have been used for testing normality. They obviously satisfy conditions (ii) and (iv) above. With regard to (i), they are possibly the simplest functions of the sample moments which satisfy the other conditions. Their numerical computation is not difficult. With regard to condition (iii), $\sqrt{\beta_1}$ and β_2 assume the values zero and 3 respectively when n' tends towards infinity.

From the work of C. C. Craig and R. A. Fisher the lower moments of $\sqrt{\beta_1}$ and β_2 for normal samples may be regarded as completely determined for all values of n'. R. A. Fisher† has given the exact values of the second, fourth and sixth Thiele semi-invariants of $\sqrt{\beta_1}$—the odd semi-invariants are of course zero, since $\sqrt{\beta_1}$ is symmetrically distributed for normal samples. These values show that for quite small values of n' the fourth and sixth semi-invariants approach closely to their normal values zero. $\sqrt{\beta_1}$ cannot, however, be regarded as an efficient test of normality. Infinite samples drawn from all symmetrical universes will give a zero value of β_1. Accordingly it is only a test of symmetry and cannot be expected to distinguish the normal from other symmetrical universes.

I have used the formulae for the B_1 and B_2 of β_2 for normal samples, which E. S. Pearson‡ has derived from R. A. Fisher's results, in the computation of the following table.

* [The P, χ^2 and P, λ_n tests have been applied, and are available. ED.]

† "The Moments of the Distribution for Normal Samples of Measures of Departure from Normality." *Proc. Royal Society of London*, Series A, Vol. CXXX. p. 22.

‡ "Notes on Tests for Normality." *Biometrika*, Vol. XXII. Parts III and IV.

TABLE A.

Values of $\sqrt{B_1}$ *and* B_2 *of* β_2 (*Normal Samples*).

Size of sample n'	$\sqrt{B_1}$	B_2
6	0·71	2·91
11	1·48	6·01
26	1·75	8·93
36	1·69	8·89
51	1·57	8·38
101	1·27	6·75

Even in samples of 101 the value of B_1 is 1·27 and the value of B_2 is 6·75, very far removed from the normal values 0 and 3 respectively. Furthermore it is evident that the approach to normality of the distribution of β_2 only improves very slowly with increasing size of sample. As the prospect of obtaining the exact or even approximate frequency distributions for β_2 for all values of n' seems remote, and as the distribution of β_2 for large samples cannot be presumed normal, it appears desirable to try to devise other tests which do not suffer from these disadvantages.

The value of the mean deviation d for infinite normal samples is

$$d = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} |x - a| \, e^{-\frac{(x-a)^2}{2\sigma^2}} \, dx = \sqrt{\frac{2}{\pi}}\,\sigma,$$

so that

$$\frac{d}{\sigma} = \sqrt{\frac{2}{\pi}} = \cdot 7978845608\ldots.$$

The test which will now be examined is that of estimating d/σ from the sample and attempting to determine the probability, or improbability, that the parent universe was normal from the difference between the estimate and $\sqrt{\frac{2}{\pi}}$.

Moments of w_n.

Suppose that, in the first place, the mean of the parent normal universe is zero. Later on the reduction of the most general case of the universal mean unknown to this case will be discussed. A random sample of n is drawn with measures $y_1, y_2, \ldots, y_n$, and the estimate of d/σ is taken to be

$$w_n = |\bar{y}|/s = \frac{\Sigma |y_i|}{ns} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

with

$$ns^2 = \Sigma y_i^2.$$

It will be evident at once that

$$\frac{1}{\sqrt{n}} \leqslant w_n \leqslant 1.$$

Furthermore the frequency distribution of w_n will obviously be independent of σ, the unknown universal standard deviation, which in consequence may be presumed unity, without loss of generality.

The probability of obtaining the given sample is then

$$\left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\Sigma y_i^2}\,\delta y_1 \ldots \delta y_n.$$

Change the variables y_i into generalised polar coordinates $(\rho;\ \phi_0, \phi_1, \ldots \phi_{n-2})$, and for brevity write s_i for $\sin\phi_i$, c_i for $\cos\phi_i$. Then

$$\left.\begin{aligned} y_1 &= \rho s_{n-2} s_{n-3} \ldots s_1 s_0 \\ y_2 &= \rho s_{n-2} s_{n-3} \ldots s_1 c_0 \\ y_3 &= \rho s_{n-2} s_{n-3} \ldots c_1 \\ &\vdots \qquad\qquad \vdots \\ y_{n-1} &= \rho s_{n-2} c_{n-3} \\ y_n &= \rho c_{n-2} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

so that $\Sigma y_i^2 = \rho^2 = ns^2$

and the integral element becomes

$$\left(\frac{1}{\sqrt{2\pi}}\right)^n \rho^{n-1} e^{-\frac{\rho^2}{2}}\,\delta\phi_0\, s_1 \delta\phi_1\, s_2^2 \delta\phi_2 \ldots s_{n-2}^{n-2}\,\delta\phi_{n-2} \quad \ldots\ldots\ldots\ldots\ldots(4),$$

from which it follows that $\rho\,(=\sqrt{n}s)$ and the $n-1$ polar angles are mutually independent. After the transformation, w_n, given by (2), is a function of the polar angles only and therefore independent of s in normal samples. Indicating as before universal mean values by square brackets, it follows that

$$[|\bar{y}|^p] = [w_n{}^p s^p] = [w_n{}^p]\,[s^p],$$

or

$$[w_n{}^p] = [|\bar{y}|^p] \div [s^p],$$

where p is any whole number, so that the moments (fixed origin) of w_n are known when the corresponding moments of $|\bar{y}|$ and s are known. Now, from (4), the frequency distribution of s is

$$Ce^{-\frac{ns^2}{2}} s^{n-1} \quad (0 \leqslant s \leqslant \infty),$$

so that the universal moments of s are as follows:

$$\left.\begin{aligned} &s_1' = [s] && s_2' = [s^2] = 1 \\ &s_3' = [s^3] = \frac{n+1}{n}\, s_1' && s_4' = [s^4] = \frac{n+2}{n} \\ &s_5' = [s^5] = \frac{(n+1)(n+3)}{n^2}\, s_1' && s_6' = [s^6] = \frac{(n+2)(n+4)}{n^2} \\ &s_7' = [s^7] = \frac{(n+1)(n+3)(n+5)}{n^3}\, s_1' && s_8' = [s^8] = \frac{(n+2)(n+4)(n+6)}{n^3} \\ &\text{etc.} \end{aligned}\right\} \ldots(5),$$

and

$$s_1' = [s] = C\int_0^\infty e^{-\frac{ns^2}{2}} s^n\, ds,$$

with $$1 = C\int_0^\infty e^{-\frac{ns^2}{2}} s^{n-1}\,ds,$$

or (for n even) $$s_1' = \frac{n!}{\{(\frac{1}{2}n)!\}^2}\frac{n^{\frac{1}{2}}}{2^n}\sqrt{\frac{\pi}{2}}.$$

But by Stirling's Theorem

$$n! = n^{n+\frac{1}{2}}\sqrt{2\pi}\,e^{-n+\frac{1}{12n}-\frac{1}{360n^3}+\frac{1}{1260n^5}-\frac{1}{1680n^7}+\cdots},$$

whence $$s_1' = e^{-\frac{1}{4n}+\frac{1}{24n^3}-\frac{1}{20n^5}+\frac{17}{112n^7}-\cdots} \qquad \text{.....(6).}$$

The universal moments of $|\bar{y}| = \frac{1}{n}\Sigma|y_i|$ may be determined very easily since

$$[|y|^p] = \sqrt{\frac{2}{\pi}}\int_0^\infty y^p e^{-\frac{y^2}{2}}\,dy$$

$$= \begin{cases} (p-1)(p-3)\ldots 3\cdot 1 \text{ when } p \text{ is even,} \\ (p-1)(p-3)\ldots 4\cdot 2\sqrt{\frac{2}{\pi}} \text{ when } p \text{ is odd,} \end{cases}$$

and since the variables y_i are mutually independent. The first eight moments, m_k', of $|\bar{y}|$ are as follows:

$$\left.\begin{aligned}
m_1' &= \frac{1}{n}[\Sigma|y_i|] = \frac{1}{n}\Sigma[|y_i|] = \sqrt{\frac{2}{\pi}} = a \\
m_2' &= \frac{1}{n^2}\{n_1 + a^2 n_2\} \\
m_3' &= \frac{a}{n^3}\{2n_1 + 3n_2 + a^2 n_3\} \\
m_4' &= \frac{1}{n^4}\{3n_1 + (8a^2+3)n_2 + 6a^2 n_3 + a^4 n_4\} \\
m_5' &= \frac{a}{n^5}\{8n_1 + 35n_2 + (20a^2+15)n_3 + 10a^2 n_4 + a^4 n_5\} \\
m_6' &= \frac{1}{n^6}\{15n_1 + (88a^2+45)n_2 + (165a^2+15)n_3 + (40a^4+45a^2)n_4 \\
&\qquad + 15a^4 n_5 + a^6 n_6\} \\
m_7' &= \frac{a}{n^7}\{48n_1 + 483n_2 + 7(64a^2+75)n_3 + 105(5a^2+1)n^4 \\
&\qquad + 35(2a^4+3a^2)n_5 + 21a^4 n_6 + a^6 n_7\} \\
m_8' &= \frac{1}{n^8}\{105n_1 + (1280a^2+735)n_2 + (4562a^2+630)n_3 \\
&\qquad + (1568a^4+2940a^2+105)n_4 + (1330a^4+420a^2)n_5 \\
&\qquad + (112a^6+210a^4)n_6 + 28a^6 n_7 + a^8 n_8\}
\end{aligned}\right\} \text{...(7).}$$

where $n_1 = n, \quad n_2 = n(n-1), \quad n_3 = n(n-1)(n-2)$, etc.

The moments, μ_k', of w_n may be written down at once from (5), (6) and (7) since

$$\mu_k' = m_k'/s_k'.$$

Finally the Thiele semi-invariants λ_k of w_n can be found from the recurrence formulae

$$\begin{aligned}
&\mu_1' = \lambda_1,\\
&\mu_2' = \lambda_1\mu_1' + \lambda_2,\\
&\mu_3' = \lambda_1\mu_2' + 2\lambda_2\mu_1' + \lambda_3,\\
&\mu_4' = \lambda_1\mu_3' + 3\lambda_2\mu_2' + 3\lambda_3\mu_1' + \lambda_4,\\
&\text{etc.}
\end{aligned}$$

On account of the intervention of the exponential in the formula for s_1', it is not possible to give simple rational expressions for the λ_k such as R. A. Fisher has given for the semi-invariants of his functions $k_3/k_2^{\frac{3}{2}}$ and k_4/k_2^2 (very similar to $\sqrt{\beta_1}$ and β_2). The calculation of the semi-invariants of w_n for any particular values of n, step by step as indicated above, is a lengthy process but presents no particular difficulty. As the value of λ_2 (equal to the square of the standard deviation) is quite small even for low values of n (see Table B below), and as λ_k appears to be of the same order of magnitude as $\lambda_2^{k/2}$, it is necessary to work to many places of decimals in order to obtain sufficiently accurate values for the higher semi-invariants. The values of the first four semi-invariants and of $\sqrt{\beta_1}$ and β_2 for certain sizes of samples are given in the following table.

TABLE B.

Semi-invariants, etc., of w_n.

Size of sample n $(=n'-1)$	λ_1	λ_2	λ_3	λ_4	$\sqrt{B_1}=\lambda_3/\lambda_2^{\frac{3}{2}}$	$B_2=\lambda_4/\lambda_2^2+3$
5	·838525	·00617082	−·00027074	·000006184	−0·56	3·16
10	·818049	·00375295	−·00010696	·000003557	−0·47	3·25
25	·805901	·00167809	−·00002250	·000000461	−0·33	3·16
50	·801884	·00086990	−·00000616	·000000072	−0·24	3·10
100	·799882	·000442786	−·00000161	·000000010	−0·17	3·05

Even for small samples, the values of $\sqrt{B_1}$ and B_2 are quite near the normal values. The values of $\lambda_p/\lambda_2^{p/2}$ for $p=3$, 4, 5 and 6 for samples of 10 are as follows, correct to two decimal places:

$$\lambda_3/\lambda_2^{\frac{3}{2}} = -0{\cdot}47, \qquad \lambda_4/\lambda_2^2 = +0{\cdot}25,$$
$$\lambda_5/\lambda_2^{\frac{5}{2}} = -0{\cdot}10, \qquad \lambda_6/\lambda_2^3 = -0{\cdot}29.$$

From this investigation it appears very likely that, for quite small samples drawn at random from a normal universe with mean zero, the distribution of w_n is fairly close to normal.

The General Case.

In the foregoing analysis it has been assumed that the universal mean of the parent normal universe is known to be zero. This case has some practical importance. It occurs when the sample variables are themselves differences of pairs of variables drawn from the same universe. In the general case if the n' random and independent observations were $x_1, x_2, \ldots x_{n'}$ we might take $y_1 = x_1 - x_2$, $y_2 = x_3 - x_4$, etc. (if n' were even), so that if the x's are normally distributed so are the y's, but with universal mean zero, and the w_n of the y's might be used to test normality. It is evident, however, that this procedure would not sufficiently utilise the information to be derived from the sample of x's.

The simplest transformation is of course

$$y_i = x_i - \bar{x} \quad (i = 1, 2, \ldots n'),$$

and in fact it is implicit in the tests $\sqrt{\beta_1}$ and β_2. It has the great advantage of being symmetrical, but cannot be used in the calculation of w_n because the y's, while normally distributed with mean zero, are not mutually independent.

The properties of an interesting transformation will now be examined*. Suppose that the n' x's are arranged in random order, for example in the order in which they were drawn. Let

$$\left.\begin{aligned}
y_1 &= \left(\frac{x_1+x_2}{2} - x_2\right)\sqrt{\frac{2}{1}} = \frac{x_1 - x_2}{\sqrt{2}} \\
y_2 &= \left(\frac{x_1+x_2+x_3}{3} - x_3\right)\sqrt{\frac{3}{2}} = \frac{x_1 + x_2 - 2x_3}{\sqrt{6}} \\
&\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
y_{n'-1} &= \left(\frac{x_1+x_2+\ldots+x_{n'}}{n'} - x_{n'}\right)\sqrt{\frac{n'}{n'-1}} = \frac{x_1+x_2+\ldots+x_{n'-1}-(n'-1)x_{n'}}{\sqrt{n'(n'-1)}} \\
y_{n'} &= \frac{x_1+x_2+\ldots+x_{n'}}{\sqrt{n'}}
\end{aligned}\right\} \quad \ldots\ldots(8).$$

The reciprocal is

$$\left.\begin{aligned}
x_1 &= \frac{y_1}{\sqrt{2}} + \frac{y_2}{\sqrt{6}} + \ldots + \frac{y_{n'-1}}{\sqrt{n'(n'-1)}} + \frac{y_{n'}}{\sqrt{n'}} \\
x_2 &= -\frac{y_1}{\sqrt{2}} + \frac{y_2}{\sqrt{6}} + \ldots + \frac{y_{n'-1}}{\sqrt{n'(n'-1)}} + \frac{y_{n'}}{\sqrt{n'}} \\
&\vdots \qquad\qquad\qquad \vdots \\
x_{n'-1} &= -y_{n'-2}\sqrt{\frac{n'-2}{n'-1}} + \frac{y_{n'-1}}{\sqrt{n'(n'-1)}} + \frac{y_{n'}}{\sqrt{n'}} \\
x_{n'} &= -y_{n'-1}\sqrt{\frac{n'-1}{n'}} + \frac{y_{n'}}{\sqrt{n'}}
\end{aligned}\right\} \quad \ldots\ldots(8').$$

* This transformation has been given by William Burnside in his *Theory of Probability* (Cambridge University Press, 1928, p. 97). Dr Karl Pearson kindly informs me that it is due to Helmert (1875). Its geometrical properties indicate its fundamental importance. For example, when $n'=4$, the four

The structure of the transformation is evident: y_1 represents the difference (to a numerical factor) between the mean of the first two observations and the second, y_2 the difference (to a numerical factor) between the mean of the first three observations and the third, etc. The transformation is completed by setting $y_{n'}$ equal to $\sqrt{n'}\,\bar{x}$. Quite apart from its mathematical implications, which will presently be examined, the method of analysis of the primary data is therefore a very natural one.

The new variables $y_1, y_2, \ldots y_{n'}$ have the following properties:

(i) They are normally distributed with the same standard deviation as the x's.

(ii) The universal mean of each of the variables $y_1, y_2, \ldots y_{n'-1}$ is zero.

(iii) Each pair of the first $n'-1$ variables are uncorrelated and accordingly, since they are normally distributed, they are independent.

(iv) The transformation is orthogonal, i.e. $\Sigma x_i^2 = \Sigma y_i^2$, or

$$\sum_{i=1}^{n'-1} y_i^2 = \sum_{i=1}^{n'} x_i^2 - n'\bar{x}^2 = ns^2$$

with $n = n' - 1$.

The n variables y_i satisfy all the conditions specified in the previous section and accordingly we take

$$w_n = \frac{\Sigma\,|y_i|}{ns}$$

as the test of normality.

The advantages of w_n, regarded as a function of the original variables x, are as follows: like β_2 it assumes a characteristic value for infinite normal random samples; the values of its semi-invariants indicate that its distribution is far closer to the normal, even for moderate samples, than that of β_2; and as will appear in the following section its frequency distribution can be determined for all normal samples. Its principal disadvantage is that it is not symmetrical in the original variables, which means that in general it will assume different values for the $n'!$ permutations of the x_i. Consequently there is a risk not present in the use of β_2 that an abnormal value of w_n may indicate an unrepresentative permutation of the sample and not a non-normal parent universe. Accordingly it would seem advisable in all cases to randomise the sample a few times and to calculate w_n for each permutation. The mean or median value of the different w_n's might then be taken as the representative value for the purpose of determining the probability of normality.

Another course meriting practical and algebraical investigation is that of taking

$$w_n' = \sum_{i=1}^{n'} |x_i - \bar{x}| \Big/ s\sqrt{n'(n'-1)} = \frac{1}{\sqrt{n'}}\,\Sigma\,|x_i - \bar{x}| \Big/ \sqrt{\Sigma\,(x_i - \bar{x})^2}$$

planes $x_1=0$, $x_2=0$, $x_3=0$, $x_4=0$ in the y-space for $y_4=$constant are the sides of the regular tetrahedron with centre of gravity at the origin, one side parallel to the plane $y_3=0$, one vertex on the y_3-axis and one edge parallel to the y_1-axis. This transformation is probably the most symmetrical orthogonal transformation of variables, one of which is $y_{n'} = \frac{1}{\sqrt{n'}}\,\Sigma x_i$, which could be devised.

instead of w_n. It may be shown that

$$[w_n'] = [w_n],$$

and, of course:

$$\lim_{n\to\infty} w_n' = \lim_{n\to\infty} w_n,$$

and w_n', a symmetrical function of the x_i, has a unique value for each sample. It is evident that for moderately sized samples the value of w_n' will not differ much from the values of w_n for the different permutations. From work at present in progress it appears that in moderately sized samples the distribution of w_n' is very similar to that of w_n. For $n' = 51$ the values of the first three moments and of B_1 for these test functions are compared in the following table:

	Mean λ_1	Standard Deviation $\sqrt{\lambda_2}$	Third Moment λ_3	$\sqrt{B_1}=\lambda_3/\lambda_2^{\frac{3}{2}}$
w'_{50}	·8019	·0292	−·0000585	−0·24
w_{50}	·8019	·0295	−·00000616	−0·24

No significance attaches to the equality of the means: they have been made so by choice of constants. To the degree of accuracy indicated, it is very likely that for samples of more than 40 the probability points for w_n shown on Table F below may be taken as applying to w_n'.

When transformation (8) has been applied to the variables x_i the distribution of

$$u_n = \bar{y}/s$$

is given by

$$C \sin^{n-2}\theta\, \delta\theta_{n-2} \qquad (\pi \geqslant \theta_{n-2} \geqslant 0)$$

with

$$u_n = \cos\theta_{n-2}.$$

This is not an efficient test of normality or even for asymmetry in the parent universe because the value of $\bar{y}$, and hence of u_n, will be small for samples drawn from any universe. u_n might, however, be used to test the "representativeness" of the permutation of the n' variables x_i: if the value of u_n is not abnormal, then the permutation is probably representative.

If the parent universe were not normal, the efficiency of the w_n test would be jeopardised if as a result of transformation the distributions of the transformed variables y_i were brought appreciably closer to normality. It is fairly evident that in general this will not occur. If α be the universal mean of the x_i, the value of the kth transformed variable y_k will tend towards $\alpha - x_{k+1}$ with increasing k. It may be shown that if the pth semi-invariant of the variables x_i is λ_p, the pth semi-invariant of the transformed variable y_k is

$$\left\{k\left(\frac{1}{\sqrt{k(k+1)}}\right)^p + \left(-\sqrt{\frac{k}{k+1}}\right)^p\right\}\lambda_p,$$

from which it follows that if the x_i's are symmetrically distributed, so are the y_k's. The character of the distribution has not been modified fundamentally by the transformation.

The Frequency Distribution of w_n. Theoretical Aspects.

The probability of obtaining a given value of

$$w_n = |\bar{y}|/s$$

is equal to 2^n times the probability of obtaining that value with positive values of the variables y_i. The frequency distribution of w_n can therefore be found by confining attention to the region

$$y_i \geqslant 0 \qquad (i = 1, 2, \ldots n)$$

which on transformation into generalised polar coordinates corresponds to the region

$$\frac{\pi}{2} \geqslant \phi_i \geqslant 0 \qquad (i = 0, 1, 2, \ldots n-2).$$

Now
$$\sqrt{n}\, w_n = c_{n-2} + s_{n-2}(c_{n-3} + s_{n-3}(c_{n-4} + \ldots$$
$$= c_{n-2} + \sqrt{n-1}\, w_{n-1} s_{n-2},$$

where, as before, c_i and s_i represent $\cos \phi_i$ and $\sin \phi_i$ respectively.

For a given value of w_n, since $w_{n-1} \leqslant 1$,

$$\sqrt{n}\, w_n \leqslant c_{n-2} + \sqrt{n-1}\, s_{n-2},$$

or, on dividing across by $\sqrt{n}$ and setting

$$\cos \alpha = \frac{1}{\sqrt{n}}, \quad \cos \beta = w_n,$$

we find
$$\cos(\phi_{n-2} - \alpha) \geqslant \cos \beta,$$

whence the limiting values of ϕ_{n-2} are given by

$$\alpha + \beta \geqslant \phi_{n-2} \geqslant \alpha - \beta.$$

It may be shown that $\alpha + \beta \geqslant \frac{\pi}{2}$, so that the condition $\alpha + \beta \geqslant \phi_{n-2}$ may be ignored.

The condition $w_{n-1} \geqslant \frac{1}{\sqrt{n-1}}$, on a similar analysis, yields the following limiting values of ϕ_{n-2},

$$\phi_{n-2} \geqslant \frac{\pi}{4} + \gamma; \quad \phi_{n-2} \leqslant \frac{\pi}{4} - \gamma,$$

with $\cos \gamma = \sqrt{\frac{n}{2}}\, w_n$. These conditions only apply when

$$\sqrt{\frac{n}{2}}\, w_n \leqslant 1,$$

or in practice for very small samples.

Since the polar angles ϕ_i are all independent for normal samples, the joint probability of w_{n-1} and ϕ_{n-2} is given by

$$C_{n-2}\, s_{n-2}^{n-2}\, \delta\phi_{n-2} f_{n-1}(w_{n-1})\, \delta w_{n-1},$$

where $f_{n-1}(w_{n-1})$ represents the frequency distribution of w_{n-1} and where the constant C_{n-2} is determined by

$$C_{n-2}\int_0^{\pi/2} s_{n-2}^{n-2}\, d\phi_{n-2} = 1.$$

Now, when ϕ_{n-2} is fixed,

$$\sqrt{n}\,\delta w_n = \sqrt{n-1}\, s_{n-2}\,\delta w_{n-1},$$

and, on substituting for δw_{n-1}, the integral element becomes

$$C_{n-2}\sqrt{\frac{n}{n-1}}\, s_{n-2}^{n-3}\,\delta\phi_{n-2} f_{n-1}(w_{n-1})\,\delta w_n.$$

The frequency distribution of w_n is accordingly found from that of w_{n-1} by the integral iteration formula

$$f_n(w_n) = C_{n-2}\sqrt{\frac{n}{n-1}}\int_0^{\cos(\alpha-\beta)} s_{n-2}^{n-4} f_{n-1}(w_{n-1})\, dc_{n-2} \quad \ldots\ldots\ldots\ldots (9),$$

with

$$w_{n-1} = \frac{\sqrt{n}\,w_n - c_{n-2}}{\sqrt{n-1}\, s_{n-2}}.$$

When $w_n \leqslant \sqrt{\frac{2}{n}}$ the value of the function $f_{n-1}(w_{n-1})$ is zero between the limits $\cos\left(\frac{\pi}{4}+\gamma\right)$ and $\cos\left(\frac{\pi}{4}-\gamma\right)$.

A limiting value of w_{n-1} regarded as a function of ϕ_{n-2} (with w_n fixed) is found by setting

$$\frac{dw_{n-1}}{d\phi_{n-2}} = 0,$$

which gives

$$c_{n-2} = \frac{1}{\sqrt{n}\, w_n}.$$

This is found to yield a minimum value of w_{n-1}; whence, on substitution and reduction

$$(n-1)\, w_{n-1}^2 \geqslant n w_n^2 - 1.$$

If $w_n^2 \geqslant p/n$, where p is an integer less than n, this condition gives

$$w^2_{n-1} \geqslant (p-1)/(n-1).$$

It will presently be shown that for $n = 3$ the frequency distribution of w_3 is represented by different functions in the intervals

$$1 \geqslant w_3 \geqslant \sqrt{\tfrac{2}{3}}$$

and

$$\sqrt{\tfrac{2}{3}} \geqslant w_3 \geqslant \sqrt{\tfrac{1}{3}}$$

respectively. From this property and from the foregoing iteration formula it may be deduced that the frequency distribution of w_n has a different functional form in each of the intervals

$$\sqrt{\frac{p}{n}} \geqslant w_n \geqslant \sqrt{\frac{p-1}{n}} \qquad (p = 2, 3, \ldots, n).$$

Form of $f_n(w_n)$ near $w_n = 1/\sqrt{n}$.

The function w_n assumes its minimum value $1/\sqrt{n}$ when $n-1$ of the variables y_i are zero, and one is not zero. Suppose that y_1 is the non-zero variable and that the other $n-1$ variables are so small that their squares are negligible. Restricting the variables to their positive values, the joint probability of $y_1, y_2, \ldots y_n$ may be written

$$\left(\sqrt{\frac{2}{\pi}}\right)^n e^{-\frac{y_1^2}{2}} \delta y_1 \delta y_2 \ldots \delta y_n,$$

as it may be assumed without loss of generality that the standard deviation of the variables is unity. Now

$$w_n = \Sigma y_i/\sqrt{n}\, y_1,$$

whence $$y_n = v_n y_1 - y_2 - \ldots - y_{n-1} \quad \text{with} \quad v_n = \sqrt{n}\, w_n - 1.$$

Change the variable y_n in the integral element so that

$$\delta y_n = y_1 \delta v_n$$

and integrate for positive values of the variables. The element of probability for v_n in the neighbourhood of $v_n = 0$ $\left(\text{i.e. } w_n = \dfrac{1}{\sqrt{n}}\right)$ is found to be

$$\left(\sqrt{\frac{2}{\pi}}\right)^n \delta v_n \int_0^\infty e^{-\frac{y_1^2}{2}} y_1 dy_1 \int_0^{v_n y_1} dy_2 \ldots \int_0^{v_n y_1 - y_2 - \ldots - y_{n-3}} dy_{n-2} \int_0^{v_n y_1 - y_2 - \ldots - y_{n-2}} dy_{n-1}$$

$$= \frac{2^{n-1}}{\pi^{n/2}} \frac{(\frac{1}{2}n-1)!}{(n-2)!} (v_n)^{n-2} \delta v_n.$$

The last expression represents the probability of v_n near $v_n = 0$ when a particular variable is presumed not zero. Since any of the n variables may be presumed not zero to give the value $\dfrac{1}{\sqrt{n}}$ of w_n the probability of v_n is n times the foregoing expression. Changing to the original variable w_n the probability of w_n in the neighbourhood of $w_n = \dfrac{1}{\sqrt{n}}$ is finally

$$\frac{2^{n-1}}{\pi^{n/2}} \frac{n^{\frac{n+1}{2}} (\frac{1}{2}n-1)!}{(n-2)!} \left(w_n - \frac{1}{\sqrt{n}}\right)^{n-2} \delta w_n.$$

Form of $f_n(w_n)$ near $w_n = 1$.

Another integral iteration formula for $f_n(w_n)$ will now be derived which, while more complicated in manner of derivation than (9), seems more amenable to algebraical treatment. As before, let y_i $(i = 1, 2, \ldots n)$ be the normal variables with mean zero and standard deviation unity. Our attention may be confined to the positive region of the y_i space. In succession transform the variables y_i orthogonally using transformation (8) and the variables z_i into generalised polar coordinates r; $\theta_0, \theta_1, \ldots \theta_{n-2}$, so that

$$z_n = \frac{\Sigma y_i}{\sqrt{n}} = r \cos \theta_{n-2}.$$

Then since the $y_i \geqslant 0$ and $ns^2 = \Sigma y_i^2 = \Sigma z_i^2 = r^2$,

$$w_n = \frac{\Sigma y_i}{ns} = \cos\theta_{n-2},$$

and the integral element of the probability of positive values of the y_i, namely,

$$\left(\sqrt{\frac{2}{\pi}}\right)^n e^{-\frac{1}{2}\Sigma y_i^2}\,\delta y_1 \ldots \delta y_n,$$

becomes $$\left(\sqrt{\frac{2}{\pi}}\right)^n e^{-\frac{r^2}{2}} r^{n-1}\,dr\,\delta\theta_0\, s_1\delta\theta_1 \ldots s_{n-3}^{n-3}\delta\theta_{n-3}\, s_{n-2}^{n-2}\delta\theta_{n-2}.$$

The segregation of the factor

$$s_{n-2}^{n-2}\delta\theta_{n-2},$$

immediately expressible in terms of w_n, has thus been effected, and the probability of θ_{n-2} may be formally set down as

$$F_{n-2}(ct_{n-2})\, s_{n-2}^{n-2}\,\delta\theta_{n-2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

(ct_{n-2} indicates $\cot\theta_{n-2}$), the function $F_{n-2}(ct_{n-2})$ being given by

$$F_{n-2}(ct_{n-2}) = K_n \int_D d\theta_0\, s_1 d\theta_1 \ldots s_{n-3}^{n-3}\, d\theta_{n-3} \quad \ldots\ldots\ldots\ldots\ldots\ldots (11),$$

with $$K_n = \left(\sqrt{\frac{2}{\pi}}\right)^n \int_0^\infty e^{-\frac{r^2}{2}} r^{n-1}\,dr = \frac{2^{n-1}}{\pi^{n/2}}\left(\tfrac{1}{2}n-1\right)!$$

and the domain of integration D, corresponding to

$$y_i \geqslant 0 \qquad (i = 1, 2, \ldots n),$$

is given by

$$s_{n-2}\left(s_{n-3}\left(\ldots s_1\left(-\frac{s_0}{\sqrt{2}}+\frac{c_0}{\sqrt{6}}\right)+\frac{c_1}{\sqrt{12}}\right)+\frac{c_2}{\sqrt{20}}\right)+\ldots+\frac{c_{n-3}}{\sqrt{n(n-1)}}\Bigg)+\frac{c_{n-2}}{\sqrt{n}} \geqslant 0,$$

$$s_{n-2}\left(s_{n-3}\left(\ldots s_1\left(-\frac{s_0}{\sqrt{2}}+\frac{c_0}{\sqrt{6}}\right)+\frac{c_1}{\sqrt{12}}\right)+\frac{c_2}{\sqrt{20}}\right)+\ldots+\frac{c_{n-3}}{\sqrt{n(n-1)}}\Bigg)+\frac{c_{n-2}}{\sqrt{n}} \geqslant 0,$$

$$\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \vdots$$

$$s_{n-2}\left(-\sqrt{\frac{n-2}{n-1}}\, s_{n-3}c_{n-4}+\frac{c_{n-3}}{\sqrt{n(n-1)}}\right)+\frac{c_{n-2}}{\sqrt{n}} \geqslant 0$$

$$-\sqrt{\frac{n-1}{n}}\, s_{n-2}c_{n-3}+\frac{c_{n-2}}{\sqrt{n}} \geqslant 0,$$

from (8′). Dividing across by the positive s_{n-2}, it will be observed that all the terms except the last of each of the inequalities are free of the angle θ_{n-2} (and hence of w_n) and the last term becomes $\frac{ct_{n-2}}{\sqrt{n}}$. This accounts for the form of (10). Furthermore it may easily be shown that for all values of the angles $\theta_0, \theta_1, \ldots \theta_{n-3}$ the *minimum* value of the left-hand terms of the inequalities, for θ_{n-2} fixed, is

$$-\sqrt{\frac{n-1}{n}}+\frac{ct_{n-2}}{\sqrt{n}},$$

so that if ct_{n-2} satisfies the inequality

$$ct_{n-2} \geqslant \sqrt{n-1} \quad \text{or} \quad c_{n-2} = w_n \geqslant \sqrt{\frac{n-1}{n}},$$

the domain of integration D becomes

$$2\pi \geqslant \theta_0 \geqslant 0,$$

$$\pi \geqslant \theta_i \geqslant 0 \qquad (i = 1, 2, \ldots (n-3)),$$

and the function $F_{n-2}(ct_{n-2})$ becomes a constant. It follows that *the probability of* $w_n = \cos\theta_{n-2}$ *when* $1 \geqslant w_n \geqslant \sqrt{\frac{n-1}{n}}$ *is given by*

$$A_{n-2} \sin^{n-2}\theta_{n-2}\delta\theta_{n-2},$$

the constant being given by

$$A_{n-2}\int_0^{\pi} \sin^{n-2}\theta\, d\theta = 2^n.$$

Adverting to the general case, it will be observed that if, after dividing all the inequalities except the last by the positive quantity $s_{n-2}s_{n-3}$, a new angle θ'_{n-3} given by

$$\frac{ct'_{n-3}}{\sqrt{n-1}} = \frac{ct_{n-3}}{\sqrt{n(n-1)}} + \frac{ct_{n-2}}{s_{n-3}\sqrt{n}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(12)$$

is introduced, the form of the resulting $n-1$ inequalities is identical with that of the original n inequalities and hence it follows that the function $F_{n-2}(ct_{n-2})$, given by (11), may be written as

$$F_{n-2}(ct_{n-2}) = \frac{K_n}{K_{n-1}}\int_{-1}^{\frac{ct_{n-2}}{\sqrt{n-1}}} F_{n-3}(ct'_{n-3})\, s_{n-3}^{n-4}\, dc_{n-3} \quad \ldots\ldots\ldots\ldots(13),$$

ct'_{n-3} being given by (12). This is the iteration formula required. It will now be utilised to determine the form of $F_{n-2}(ct_{n-2})$, and hence the frequency of w_n when $\sqrt{\frac{n-1}{n}} \geqslant w_n \geqslant \sqrt{\frac{n-2}{n}}$; to prove in fact that the derivative

$$F'_{n-2}(ct_{n-2}) = H_{n-2}\left(1 - \frac{ct^2_{n-2}}{n-1}\right)^{\frac{n-4}{2}},$$

H_{n-2} being a constant. This will be proved by induction. Let it be assumed that the result holds for $n-1$, i.e. that for

$$\sqrt{\frac{n-2}{n-1}} \geqslant c'_{n-3} \geqslant \sqrt{\frac{n-3}{n-1}},$$

$$F'_{n-3}(ct'_{n-3}) = H_{n-3}\left(1 - \frac{ct'^2_{n-3}}{n-2}\right)^{\frac{n-5}{2}}.$$

Now it may be proved that if $w_n = c_{n-2} \geqslant \sqrt{\frac{n-2}{n}}$, then it follows that

$$ct'_{n-3} \geqslant \sqrt{\frac{n-3}{2}},$$

which is equivalent to $c'_{n-3} \geqslant \sqrt{\dfrac{n-3}{n-1}}$. Also the function $F_{n-3}(ct'_{n-3})$ is equal to a constant A_{n-3} when $c'_{n-3} \geqslant \sqrt{\dfrac{n-2}{n-1}}$, i.e. when $ct'_{n-3} \geqslant \sqrt{n-2}$. The last inequality yields the following limits for c_{n-3},

$$c_{n-3} \leqslant \cos(\alpha+\beta),$$

$$c_{n-3} \geqslant \cos(\alpha-\beta),$$

with $\cos\alpha = -1/(n-1)$ and $\cos\beta = ct_{n-2}/\sqrt{n-1}$.

Hence, from (13),

$$\frac{K_{n-1}}{K_n} F_{n-2}(ct_{n-2}) = A_{n-3} \int_{-1}^{\frac{ct_{n-2}}{\sqrt{n-1}}} s_{n-3}^{n-4}\, dc_{n-3} + \int_{\cos(\alpha+\beta)}^{\cos(\alpha-\beta)} \{F_{n-3}(ct'_{n-3}) - A_{n-3}\}\, s_{n-3}^{n-4}\, dc_{n-3}.$$

Differentiating across with respect to ct_{n-2} and noting that the integral element of the second integral vanishes at the limits of integration, we find

$$\frac{K_{n-1}}{K_n} F'_{n-2}(ct_{n-2}) = \frac{A_{n-3}}{\sqrt{n-1}} \left(1 - \frac{ct^2_{n-2}}{n-1}\right)^{\frac{n-4}{2}} + K_{n-3} \int_{\cos(\alpha+\beta)}^{\cos(\alpha-\beta)} \left(1 - \frac{ct'^2_{n-3}}{n-2}\right)^{\frac{n-5}{2}} s_{n-3}^{n-3}\, dc_{n-3}.$$

On substitution for ct'_{n-3} in the latter integral, upon reduction it is found to be of the form (to a constant factor)

$$\int_{x_0}^{x_1} \{(x-x_0)(x-x_1)\}^{\frac{n-5}{2}}\, dx,$$

which (to a constant factor) is equal to

$$(x_1 - x_0)^{n-4}.$$

In this case

$$x_1 - x_0 = \cos(\alpha-\beta) - \cos(\alpha+\beta) = 2\sin\alpha\sin\beta$$

$$= 2\sqrt{\frac{n(n-2)}{(n-1)^2}} \left(1 - \frac{ct^2_{n-2}}{n-1}\right)^{\frac{1}{2}}.$$

It follows that (to a constant factor) $F'_{n-2}(ct_{n-2})$ is equal to

$$\left(1 - \frac{ct^2_{n-2}}{n-1}\right)^{\frac{n-4}{2}}.$$

All the constants have been determined, and the final result is as follows: *the probability of* $w_n = \cos\theta$ *when* $\sqrt{\dfrac{n-1}{n}} \geqslant w_n \geqslant \sqrt{\dfrac{n-2}{n}}$ *is*

$$\left\{\frac{2^{n-1}}{\pi} \frac{n(n-2)}{\sqrt{n-1}} \int_0^{ct_{n-2}} \left(1 - \frac{x^2}{n-1}\right)^{\frac{n-4}{2}} dx - \frac{n-2}{2} A_{n-2}\right\} \sin^{n-2}\theta\delta\theta,$$

with $$A_{n-2} \int_0^{\pi} \sin^{n-2}\theta d\theta = 2^n.$$

Frequency Distributions for $n = 2, 3, 4$ and 5.

The foregoing general propositions completely determine the probabilities of w_n for $n = 2$ and $n = 3$ as follows:

$n = 2.$

$$1 \geqslant w_2 \geqslant \frac{1}{\sqrt{2}} \; : \; \frac{4}{\pi} \frac{\delta w_2}{\sqrt{1 - w_2^2}}.$$

$n = 3.$

$$1 \geqslant w_3 \geqslant \sqrt{\tfrac{2}{3}} \; : \; 4\delta w_3,$$

$$\sqrt{\tfrac{2}{3}} \geqslant w_3 \geqslant \sqrt{\tfrac{1}{3}} \; : \; 4\left(1 - \frac{3}{\pi} \cos^{-1} \frac{w_3}{\sqrt{2 - 2w_3^2}}\right) \delta w_3.$$

$n = 4.$

The probabilities of w_4 when $w_4 \geqslant \dfrac{1}{\sqrt{2}}$ are as follows:

$$1 \geqslant w_4 \geqslant \sqrt{\tfrac{3}{4}} \; : \; \frac{32}{\pi} \sin\theta \delta w_4,$$

$$\sqrt{\tfrac{3}{4}} \geqslant w_4 \geqslant \sqrt{\tfrac{2}{4}} \; : \; \frac{32}{\pi}\left(\frac{2}{\sqrt{3}} \cos\theta - \sin\theta\right) \delta w_4,$$

with $w_4 = \sin\theta$. The form of the probability function has also been determined for the third interval, namely $\sqrt{\tfrac{2}{4}} \geqslant w_n \geqslant \sqrt{\tfrac{1}{4}}$, but perhaps the details need not be given. The first derivative of $F_2(ct_2)$ was found to be

$$F_2'(ct_2) = \frac{64}{\sqrt{3}\pi}\left(1 - \frac{3}{\pi} \cos^{-1} \sqrt{\frac{2ct_2^2}{3 - ct_2^2}}\right).$$

(Compare with the frequency distribution of w_3 in the second interval.) It was found possible to integrate this expression so as to give the following expression for the probability of w_4 in the interval

$$\sqrt{\tfrac{3}{4}} \geqslant w_4 \geqslant \sqrt{\tfrac{1}{4}} \; : \; \frac{32}{\pi}\left(\frac{2}{\sqrt{3}} \cos\theta - \sin\theta - 2\sqrt{3} \cos\theta \cos^{-1} \sqrt{\frac{2\cot^2\theta}{3 - \cot^2\theta}} + 6\cos^{-1} \sqrt{\frac{2}{3 - \cot^2\theta}}\right) \delta w_4,$$

with $\cos\theta = w_4$.

$n = 5.$

The probabilities of w_5 in the first two intervals are as follows:

$$1 \geqslant w_5 \geqslant \sqrt{\tfrac{4}{5}} \; : \; 24\,(1 - w_5^2)\,\delta w_5,$$

$$\sqrt{\tfrac{4}{5}} \geqslant w_5 \geqslant \sqrt{\tfrac{3}{5}} \; : \; \frac{12}{\pi}\left(2\pi - 10\cos^{-1}\frac{\cot\theta}{2} + 5\cot\theta\sqrt{1 - \frac{\cot^2\theta}{4}}\right) \sin^2\theta \delta w_5,$$

with $\cos\theta = w_5$.

It seems not unlikely that further investigation would have given more symmetrical expression to the iteration and other formulae in this section, and that exact values for the frequencies for some values of $n > 4$ and some intervals

of $w_n \leqslant \sqrt{\frac{n-2}{n}}$ might have been obtained. The complicated expressions obtained even for $n = 4$ and $n = 5$ suggest however that such formulae would be exceedingly cumbersome. The present paper has the practical objective of deriving the actual numerical probabilities of w_n for samples of different sizes, and it will be shown that the results so far obtained are adequate for this purpose

Computation of Frequency Tables for w_n.

In Table C will be found the ordinates, and in Table D the cumulative probabilities of w_n for $n = 2$ to $n = 10$ inclusive for certain values of w_n. These values were more or less arbitrarily selected, the pivotal point $w_n = \cdot 80$ as very close to the mean $\sqrt{\frac{2}{\pi}}$ for infinite samples, and the interval of $\cdot 03$ as about one-half of the standard deviation for $n = 10$ and therefore less than half the standard deviation for $n < 10$. The values of the ordinates for $n = 2$, 3 and 4 and for $w_n > \sqrt{\frac{n-2}{n}}$ are the actual values calculated from the formulae given in the last section. The remaining ordinates were computed by approximate integration from the integral iteration formula (9). For instance, in the calculation of the ordinates for the different values of w_5, the integral element on the right-hand side was regarded as a function of $\cos \phi_3$ (i.e. c_3) and its values found at intervals of $\cdot 05$ starting with $\cos \phi_3 = 0$. The values of $f_4(w_4)$ were found by osculatory interpolation* (used in life table construction) from the previously ascertained values of $f_4(\cdot 86)$, $f_4(\cdot 83)$, etc.; the values of $f_4(w_4)$ for $w_4 > \sqrt{\frac{3}{4}}$ were the actual values which were easily calculated.

Finally the values of the ordinate for each selected value of w_5 (i.e. $\cdot 47$, $\cdot 50$, etc.) were found by approximate integration by dividing the values of the integral element into contiguous groups of 3, 4 or 5 and applying the familiar formulae to each group. Special methods were used near the upper limits of $\cos \phi_3$ and near the limits of the regions where $f_4(w_4)$ is zero, as at these limits the values of $\cos \phi_3$ were not multiples of $\cdot 05$. As a check on the work the value of the integral

$$\int_{\sqrt{\frac{1}{5}}}^{1} f_5(w_5)\, dw_5$$

was computed by approximate integration and found to equal 0·999, which was sufficiently close to unity.

Without adjusting the ordinates to allow for this slight discrepancy the calculation of the ordinates for the frequency of w_6 was proceeded with in an analogous manner, and so on to $n = 10$. At each stage the proximity of the total probability to unity was checked: the actual totals ranged from 0·995 to 1·001 (without any adjustment). The work, which involves about 1000 interpolations, was arduous, but systematic.

* See Supplement to the *Seventy-fifth Annual Report of the Registrar-General of Births, Deaths and Marriages in England and Wales*, Part I—Life Tables, p. 50.

TABLE C.

Ordinates of the Frequency Distribution of w_n.

w_n \ n' / n	3 / 2	4 / 3	5 / 4	6 / 5	7 / 6	8 / 7	9 / 8	10 / 9	11 / 10
1·00	∞	4	0	0	0	0	0	0	0
·98	6·3983	4	2·0270	·9504	·4275	·1882	·0814	·0348	·0147
·95	4·0776	4	3·1806	2·3400	1·6531	1·1408	·7740	·5190	·3450
·92	3·2487	4	3·9920	3·6864	3·2698	2·8246	2·3794	1·9780	1·6315
·89	2·7924	4	4·6444	4·9340	4·7941	4·6066	4·3849	4·1346	3·88
·86	2·4951	4	4·9178	5·1070	5·4093	5·6479	5·79	5·86	5·92
·83	2·2828	4	4·0809	4·6789	5·2092	5·63	5·92	6·29	6·62
·80	2·1221	2·7019	3·2978	3·9074	4·34	4·74	5·13	5·47	5·82
·77	1·9955	1·9052	2·5574	2·91	3·21	3·52	3·75	3·98	4·20
·74	1·8930	1·4049	1·8525	1·99	2·17	2·30	2·40	2·50	2·57
·71	1·8081	1·0316	1·1779	1·28	1·32	1·36	1·38	1·37	1·37
·68	—	·7320	·7135	·777	·767	·740	·707	·678	·643
·65	—	·4810	·4362	·403	·396	·370	·336	·302	·270
·62	—	·2647	·2517	·224	·201	·168	·144	·123	·102
·59	—	·0745	·1296	·112	·0930	·0728	·0568	·0446	·0349
·56	—	—	·0532	·0494	·0370	·0294	·0216	·0154	·0104
·53	—	—	·0124	·0181	·0122	·0091	·0069	·0036	·0024
·50	—	—	0	·0040	·0035	·0023	.0016	·0007	·0004
·47	—	—	—	·0004	·0007	·0005	·0003	·0002	·0001
·44	—	—	—	—	·0000	·0001	·0000	·0000	·0000
(Lower limit of w_n)	(·7071)	(·5773)	(·5000)	(·4472)	(·4082)	(·3780)	(·3536)	(·3333)	(·3162)

In Table D the cumulative probabilities shown for $n=2$ and $n=3$ and for $w_n \geqslant \sqrt{\frac{n-1}{n}}$ are the actual values calculated from the formulae in the last section. The remainder of the table was computed by approximate integration, the formula used being based upon that for osculatory interpolation, i.e., for unit interval,

$$\int_0^1 u_{1+x}\,dx = \tfrac{1}{24}(-u_0 + 13(u_1+u_2) - u_3),$$

a formula of some interest in itself. The slight discrepancies in the total probabilities were distributed proportionately over the calculated frequencies.

As a final check, it was calculated from Table D that the mean value of w_{10}, depending for its accuracy on all the interpolations from $n=5$ to $n=10$, was 0·8177 compared with the actual value of 0·8180 given in Table B.

From the values of $\sqrt{B_1}$ and B_2 given in the latter table it might have been anticipated that the Second Approximation to the Law of Error, namely

$$\frac{1}{\sigma\sqrt{2\pi}}\left\{1 - \frac{\sqrt{B_1}}{6}\left(\frac{3w}{\sigma} - \frac{w^3}{\sigma^3}\right)\right\} e^{-\frac{w^2}{2\sigma^2}}\delta w$$

TABLE D.

Cumulative Probability of w_n from $w_n = 1$.

w_n \ n' / n	3 / 2	4 / 3	5 / 4	6 / 5	7 / 6	8 / 7	9 / 8	10 / 9	11 / 10
1·00	0	0	0	0	0	0	0	0	0
·98	·2551	·0800	·0271	·0095	·0034	·0013	·0005	·0002	·0001
·95	·4043	·2000	·1066	·0590	·0335	·0193	·0112	·0066	·0039
·92	·5128	·3200	·2147	·1495	·1071	·0777	·0578	·0420	·0314
·89	·6028	·4400	·3445	·2803	·2295	·1900	·1589	·1335	·1133
·86	·6819	·5600	·4915	·4330	·3850	·3460	·3141	·2857	·2623
·83	·7534	·6800	·6263	·5810	·5464	·5175	·4928	·4712	·4539
·80	·8193	·7856	·7369	·7106	·6909	·6745	·6608	·6500	·6433
·77	·8810	·8535	·8246	·8130	·8043	·7988	·7950	·7926	·7908
·74	·9393	·9027	·8907	·8860	·8846	·8858	·8872	·8894	·8920
·71	·9948	·9390	·9361	·9346	·9364	·9399	·9431	·9466	·9500
·68	—	·9653	·9639	·9651	·9672	·9707	·9736	·9765	·9791
·65	—	·9834	·9807	·9824	·9842	·9868	·9887	·9905	·9921
·62	—	·9945	·9908	·9915	·9929	·9945	·9956	·9966	·9973
·59	—	·9993	·9964	·9964	·9971	·9979	·9984	·9989	·9992
·56	—	—	·9990	·9987	·9990	·9993	·9995	·9997	·9998
·53	—	—	·9999	·9997	·9997	·9998	·9999	·9999	1·0000
·50	—	—	1·0000	·9999	·9999	1·0000	1·0000	1·0000	1·0000
·47	—	—	—	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000
(Lower Limit of w_n)	(·7071)	(·5773)	(·5000)	(·4472)	(·4082)	(·3780)	(·3536)	(·3333)	(·3162)

(σ here indicating the standard deviation of w), would furnish a close approximation to the actual distribution. It is extremely fortunate and somewhat unexpected that, as shown in the diagram, the correspondence is very satisfactory for samples of only 10. On this diagram the normal curve, with the actual mean and standard deviation of w_{10}, is shown for the purposes of contrast and comparison. Circles (⊙) only mark the curve for the Second Approximation, as the ordinates of the latter curve are too close to those of the actual distribution to permit of accurate drawing.

The correspondence between the Second Approximation and the actual distribution is brought out concisely in Table E, p. 330, which shows the 1 % and 5 % probability points for the respective distributions. [By definition, the upper 1 % probability point is the value of w_{10} above which lies 1 % of the total frequency.]

There is scarcely any significant difference between the probability points as calculated by these two systems. While Table B shows that the value of B_2 does not decrease continuously with increasing size of sample, it does so after a certain value of n and never differs much from the normal value of 3. There can be no

FREQUENCY DISTRIBUTION OF W_{10}

BROKEN LINE INDICATES NORMAL CURVE & CIRCLES (⊙) SHOW ORDINATES OF SECOND APPROXIMATION TO LAW OF ERROR.

W_{10}	ACTUAL	SECOND APPROX L OF E	NORMAL CURVE
1·00	0·00	-0·03	0·08
·98	0·01	0·04	0·20
·95	0·35	0·47	0·64
·92	1·63	1·68	1·63
·89	3·88	3·75	3·27
·86	5·92	5·84	5·15
·83	6·62	6·68	6·39
·80	5·82	5·82	6·24
·77	4·20	4·09	4·79
·74	2·57	2·50	2·89
·71	1·37	1·40	1·37
·68	0·64	0·70	0·51
·65	0·27	0·30	0·15
·62	0·10	0·10	0·04
·59	0·03	0·03	0·01
·56	0·01	0·01	0·00

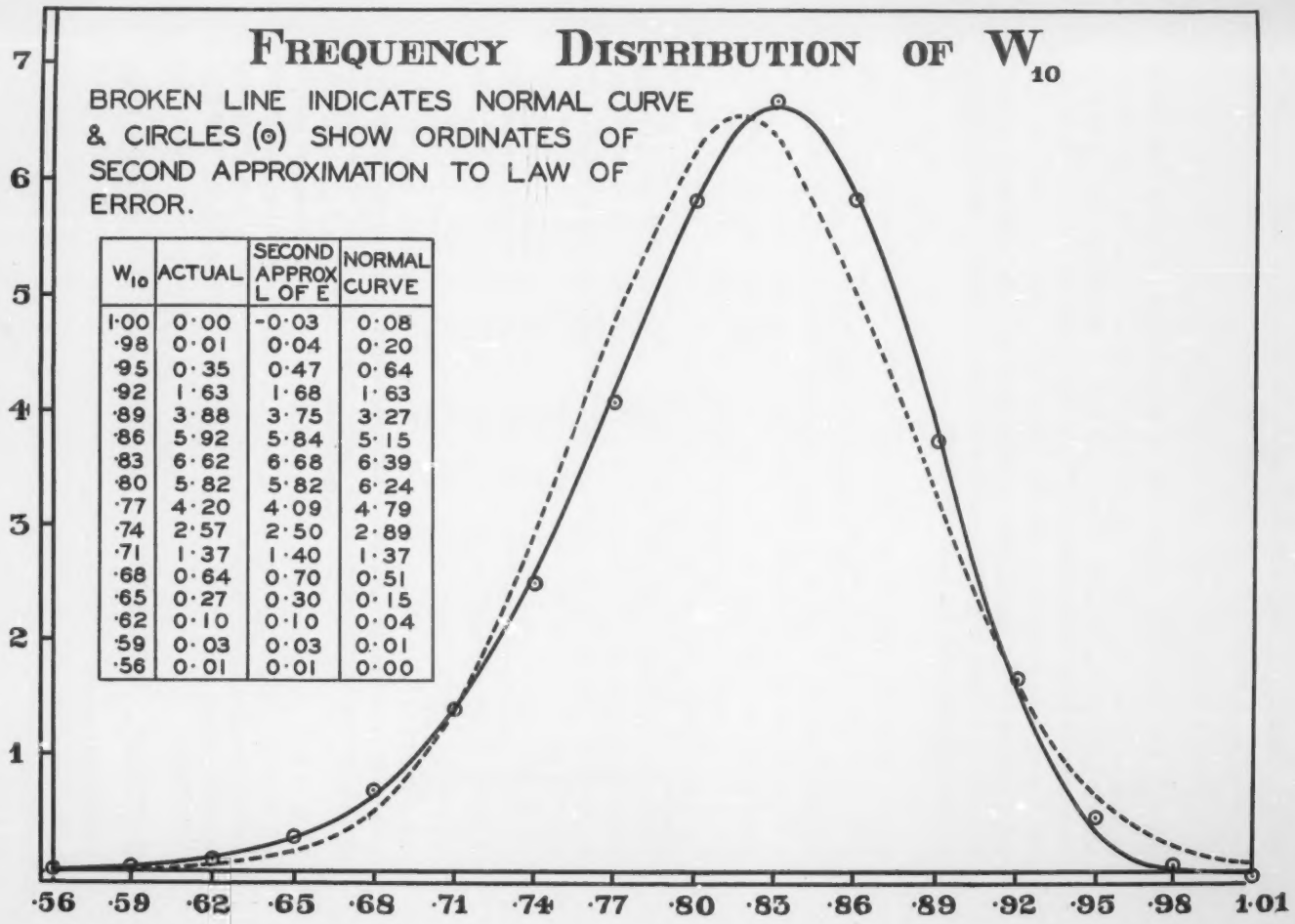


TABLE E.

	1 % points		5 % points	
	Upper	Lower	Upper	Lower
Actual distribution	·941	·656	·911	·710
Second Approximation to Law of Error	·942	·656	·912	·708

reasonable doubt that the Second Approximation will yield sufficiently accurate 1 % and 5 % probability points for samples of 11 ($=n'$) and over. On this assumption the following table has been calculated:

TABLE F.

The 1 % and 5 % probability points of w_n.

Size of Sample n'	n	Upper limit of w_n	Probability points of w_n				Lower limit of w_n	Mean of w_n	Standard Deviation of w_n
			Upper 1 %	Upper 5 %	Lower 5 %	Lower 1 %			
6	5	1	·980	·954	·696	·626	·4472	·8385	·0786
11	10	1	·941	·911	·710	·656	·3162	·8180	·0613
16	15	1	·916	·891	·720	·677	·2581	·8113	·0516
21	20	1	·902	·879	·728	·691	·2236	·8079	·0454
26	25	1	·892	·870	·734	·701	·2000	·8059	·0410
31	30	1	·884	·864	·739	·709	·1826	·8046	·0376
36	35	1	·878	·859	·743	·715	·1690	·8036	·0350
41	40	1	·873	·855	·746	·720	·1581	·8029	·0328
46	45	1	·869	·851	·749	·725	·1491	·8023	·0310
51	50	1	·865	·849	·751	·728	·1414	·8019	·0295
76	75	1	·853	·839	·759	·741	·1155	·8005	·0242
101	100	1	·846	·834	·764	·748	·1000	·7999	·0210
501	500	1	·820	·814	·783	·776	·0447	·7983	·0095
1001	1000	1	·813	·809	·787	·782	·0316	·7981	·0067

Conclusion.

It is a simple matter to devise theoretical universes which random samples will identify as probably non-normal by the w_n test and to determine roughly from Table F the sizes of the samples required. For example, the value of the ratio of mean deviation to standard deviation (in infinite random samples) for the rectangular distribution

$$\tfrac{1}{2}\delta x \qquad (+1 \geqslant x \geqslant -1)$$

is $\frac{\sqrt{3}}{2} = ·866$, so that, with no presumption of knowledge about the nature of the distribution, samples of about 75 might be required in order that in the majority

of cases the values of w_n found would lie above the upper 1% probability point. The values of the ratios (in infinite samples) for the distributions

$$\frac{2}{\pi}\frac{\delta x}{(1+x^2)^c} \text{ and } \tfrac{1}{2}e^{-|x|}\delta x \qquad (+\infty \geqslant x \geqslant -\infty)$$

which bear superficial resemblances to the normal, are $\frac{2}{\pi} = \cdot 637$ and $\frac{1}{\sqrt{2}} = \cdot 707$ respectively, and samples of about 25 and 50 respectively would usually be required to indicate their non-normality. It is important to observe, however, that the test will often react for far smaller samples. These three universes are symmetrical. In cases where asymmetry is suspected a count of + signs and − signs of the transformed variables y_i in moderate sized samples (say of 50 to 100) may reveal such asymmetry. The number of signs should of course be distributed in the "point binomial"

$$(\tfrac{1}{2} + \tfrac{1}{2})^n$$

for normal samples.

The efficiency of w_n for testing normality with the types of samples actually occurring is now a matter for practical investigation. The indications are that *in practice* it will not often react for samples of less than, say, 50. In using it we are manifestly liable to what J. Neyman and E. S. Pearson* have called "Errors of Type II," i.e. of accepting an hypothesis H_0 (in this case that the universe is normal) when some alternative is true. The question arises: if w_n fails to identify non-normality for a sample known to be derived from a non-normal universe will any other *general* method prove effective? In other words, an "existence theorem" for tests of normality seems to be required: a theorem which, while not necessarily indicating the form of the test or tests, will at least establish the "maximum sensitivity" (with a suitable definition of "sensitivity") however approximately and with whatever simplification in the conditions of the problem. This may be quite small for small samples; it is manifestly nil for samples of 2. As shown by R. A. Fisher† for $\sqrt{\beta_1}$ in normal samples of 3 and in the present paper for w_2, the distributions are wholly or partially U-shaped, and in consequence they reveal no region of improbability. On the other hand w_n must identify non-normality in indefinitely large samples, unless the universe has the normal value of $\sqrt{\frac{2}{\pi}}$ for the ratio of mean deviation to standard deviation. The only important non-normal distribution which assumes the normal value for the ratio appears to be the Second Approximation to the Law of Error which also gives the normal value of 3 for β_2 in infinite samples.

Summary.

The values of the moments and of $\sqrt{B_1}$ and B_2 of the distribution of β_2 for normal samples indicate that impracticably large samples would be required in order that the distribution would be presumed normal. As the frequency distribution of β_2 is unknown and as $\sqrt{\beta_1}$ is to be regarded as a test of asymmetry rather

* "The Testing of Statistical Hypotheses in Relation to Probabilities *a priori*." *Proceedings of the Cambridge Philosophical Society*, Vol. XXIX. Part 4, p. 493.

† *Op. cit.*, p. 20.

than of normality, we are constrained to seek new measures of normality. The measure considered in this paper is the ratio of the mean deviation to the standard deviation calculated from the sample, which ratio in infinite random samples equals $\sqrt{\frac{2}{\pi}}$.

Two cases are considered: (i) that in which the experimental technique is such that the universal mean may be presumed known and hence, by a change of origin, presumed zero; (ii) the general case of both universal mean and standard deviation unknown. In the first case, if the sample measures are $y_1, y_2, \ldots y_n$, we take

$$w_n = \frac{1}{\sqrt{n}} \frac{\Sigma |y_i|}{\sqrt{\Sigma y_i^2}}.$$

In the second case, if the sample measures are $x_1, x_2, \ldots x_{n'}$, arranged in random order, we take

$$w_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{|x_1 + \ldots + x_i - i x_{i+1}|}{\sqrt{i(i+1)}} \Big/ \sqrt{\sum_{i=1}^{n'} (x_i - \bar{x})^2},$$
$$n = n' - 1.$$

In this case w_n has not an unique value for each sample and it is suggested tentatively (pending further work) that for moderately sized samples the function

$$w_n' = \frac{1}{\sqrt{n'}} \sum_{i=1}^{n'} |x_i - \bar{x}| \Big/ \sqrt{\sum_{i=1}^{n'} (x_i - \bar{x})^2}$$

be used instead of w_n. The frequency distribution of w_n may be regarded as completely determined for all normal samples. From Table F may be determined the approximate improbability of the value of w_n found from a given sample occurring if the sample were drawn at random from a normal universe.

II.

A COMPARISON OF β_2 AND MR GEARY'S w_n CRITERIA.

By E. S. PEARSON, D.Sc.

1. *The Choice of alternative Tests.*

In the development of statistical sampling theory it has often happened that more than one test of a given hypothesis is available. Generally on theoretical grounds it is possible to specify which of these tests is the most efficient; but it may happen that owing to mathematical difficulties in putting the ideal test into working form or to practical difficulties arising from the extent of computation involved, the statistician will choose to employ a second best but simpler test. In the case of testing the hypothesis that a sample has been drawn from a normally distributed population, it seems likely that for large samples and when only small departures from normality are in question, the most efficient criteria will be based on the moment coefficients of the sample, e.g. on the values of $\sqrt{\beta_1}$ and β_2*. While, however, the sampling moments of these quantities are known*, it has not so far been possible to determine the actual sampling distributions from which the probability integral could be obtained exactly.

In earlier papers in this Journal† I suggested that empirical curves having the correct first four moment coefficients might be used as approximations to the unknown true distributions, and gave tables of 5 °/₀ and 1 °/₀ levels of probability for $\sqrt{\beta_1}$ and β_2, which have since been republished in *Tables for Statisticians and Biometricians*, Part II (Table XXXVII *bis*). The sampling distribution of $\sqrt{\beta_1}$ being symmetrical and not very leptokurtic presents relatively little difficulty, and a recent investigation by K. Williams‡ suggests that the tables previously given cannot be far in error. A test on $\sqrt{\beta_1}$ is however only sensitive to lack of symmetry, and the adequacy of the tables for the 5 °/₀ and 1 °/₀ limits for β_2 is at present more doubtful, since the moment coefficients indicate a very skew and leptokurtic distribution which it is difficult to approximate to with confidence from a knowledge of four moment coefficients only.

R. C. Geary in his interesting paper above has therefore suggested an alternative criterion, which is based on the ratio of the mean deviation to the standard

* R. A. Fisher, *Proc. Roy. Soc.* Series A, Vol. 130 (1930), pp. 16—28.

† *Biometrika*, Vol. XXII. (1930), pp. 239—249, and pp. 423—424.

‡ *Biometrika*, Vol. XXVII. (1935), pp. 269—271. Mr Williams has extended my tables for $\sqrt{\beta_1}$ down to $n'=20$.

deviation. This criterion which, as shown below, is sensitive to changes in the population β_2 has the advantage that

(1) it yields a more manageable sampling distribution, for which he has been able to provide tables of percentage limits;

(2) it involves less trouble in calculation, since the mean deviation is more readily obtained than the fourth moment.

In assessing the value of these alternative tests it is necessary to bear in mind two risks of error in judgment:

Type I, that of rejecting the hypothesis tested when it is true, i.e. of deciding the population is not normal when it is;

Type II, that of failing to detect departure from normality when it really exists in the population.

In the case of β_2, no attempt was made in my previous paper to calculate probability levels for $n < 100$ because of the very high values of $B_1(\beta_2)$ and $B_2(\beta_2)$, and before further investigation is completed, the limits given for samples of less than 200 or 300 must be regarded as only approximate. Mr Geary has however been able to obtain precise levels for his ratio w_n and reports that investigations at present in progress give reasons for believing that for $n' \geqslant 40$ the more generally useful criterion w_n' will have the same sampling distribution as w_n. His test therefore as far as it concerns kurtosis rather than asymmetry does provide a reliable control of the error of Type I. Its efficiency in dealing with the other type of error can be most readily judged on the basis of some numerical investigations. Such investigations have been carried out after consultation with Mr Geary*; the work has been rather lengthy as it seemed desirable to compute values for $\sqrt{\beta_1}$ and β_2 as well as for w_n', and in some cases w_n.

2. *The Experimental Sampling.*

The problem to be dealt with is that of comparing the relative efficiency of w_n, w_n', $\sqrt{\beta_1}$ and β_2 in controlling errors of Type II, i.e. in detecting departure from normality in the sampled population. Samples were drawn with the help of Tippett's Random Sampling Numbers† from the seven different populations, which are shown in Table I. Further details of the experiment are given in an Appendix to this paper.

The populations II, III, IV, VI and VII have been used in a number of previous sampling experiments‡; I and V, the rectangular and double exponential populations are referred to by Geary on p. 330 of his paper. Population IV was derived from the equation

$$y = y_0(1 + x^2)^{-3} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

* My apologies are due to Mr Geary for the delay in the publication of his paper which has resulted, but I hope that he will regard the work as worth while. The heavy computation has been undertaken by students in the Department of Applied Statistics, University College, London, and in particular by Miss C. M. Thompson.

† *Tracts for Computers*, No. XV.

‡ For an account of these distributions see also J. M. le Roux, *Biometrika*, Vol. XXIII. (1931), pp. 158—161.

When the distribution law is very leptokurtic there is inevitably a certain difference between the moments of the probability distribution represented by the Random Number Scale (adjusted to a total of 10,000) used in sampling and the moments of the theoretical law. It is the former that have been entered in Table I. For example, for the double exponential curve $\beta_2 = 6{\cdot}0$ and (Mean deviation)/(Standard deviation) $= w_n = {\cdot}707$; but for the sampling scale used $\beta_2 = 5{\cdot}88$ and $w_n = {\cdot}700$.

TABLE I.

Populations sampled.

No.	Type	$\sqrt{\beta_1}$	β_2	Mean deviation / Standard deviation
I	Rectangular ...	0·0	1·8	·866
II	Pearson Type II	0·0	2·5	·818
III	" " VII	0·0	4·1	·768
IV	" " VII	0·0	7·1	·734
V	Double Exponential	0·0	5·9	·700
VI	Pearson Type III	0·7	3·75	·789
VII	" " I	1·0	3·8	·797

Again the law of equation (1), p. 334, above has $\beta_2 = 9{\cdot}0$, while for the sampling scale $\beta_2 = 7{\cdot}05$; the difference arises owing to the practical difficulty of representing adequately for sampling purposes a curve of this form. These points are referred to again in the Appendix.

The sampling may be divided into three groups.

Series A.

Six samples of $n' = 50$* were drawn from each of the populations II, III, VI and VII. If the 50 values of the variable x are denoted by $x_1, x_2, \ldots x_{50}$, then Geary's criterion w_n may be calculated from the $n = n' - 1 = 49$ values of y obtained by the transformation of his equation (8). By random re-arrangements of the 50 x's, four series of y's were obtained, and hence four different values of w_n, for each of the six samples from the four populations. Samples of 50 were also obtained from a Normal population and treated in the same manner. Each series of 50 values of x gave a single value of

$$w_n' = \frac{\sum_{i=1}^{n'} |x_i - \bar{x}|}{\sqrt{n' \sum_{i=1}^{n'} (x_i - \bar{x})^2}} = \frac{\text{Mean deviation}}{\text{Standard deviation}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

where $\bar{x}$ is the mean value of x in the sample. Table II contains all these values of w_n and w_n', which have also been plotted in Fig. 1. On this diagram the lower

* n' will be used for the sample size to conform with Geary's notation.

TABLE II.

Samples of 50; *Values of* w_n *and* w_n'.

Sample	Normal Population		Population II		Population III		Population VI		Population VII	
	w_n	w_n'	w_n	w_n'	w_n	w_n'	w_n	w_n'	w_n	w_n'
1 *a*	·791	·802	·856	·864	·760	·769	·822	·822	·764	·778
1 *b*	·820		·831		·755		·813		·791	
1 *c*	·813		·850		·780		·833		·782	
1 *d*	·800		·842		·761		·809		·775	
2 *a*	·799	·822	·859	·854	·791	·774	·792	·792	·832	·805
2 *b*	·780		·859		·797		·815		·817	
2 *c*	·838		·864		·762		·800		·737	
2 *d*	·834		·861		·756		·817		·830	
3 *a*	·774	·749	·810	·814	·827	·811	·830	·796	·830	·817
3 *b*	·754		·820		·799		·798		·826	
3 *c*	·759		·788		·788		·783		·807	
3 *d*	·758		·816		·818		·794		·833	
4 *a*	·817	·825	·808	·790	·786	·789	·821	·814	·800	·788
4 *b*	·827		·809		·809		·818		·791	
4 *c*	·833		·807		·778		·784		·817	
4 *d*	·830		·830		·785		·795		·774	
5 *a*	·824	·826	·782	·816	·786	·782	·789	·787	·833	·812
5 *b*	·812		·788		·767		·808		·800	
5 *c*	·796		·824		·783		·792		·812	
5 *d*	·817		·795		·787		·808		·767	
6 *a*	·801	·807	·835	·841	·758	·725	·811	·847	·828	·806
6 *b*	·800		·836		·748		·847		·823	
6 *c*	·804		·834		·720		·859		·776	
6 *d*	·827		·848		·754		·868		·800	
Mean	·8045	·805	·8272	·830	·7773	·775	·8127	·810	·8019	·801

N.B. *a*, *b*, *c* and *d* are four random arrangements of the same 50 observations.

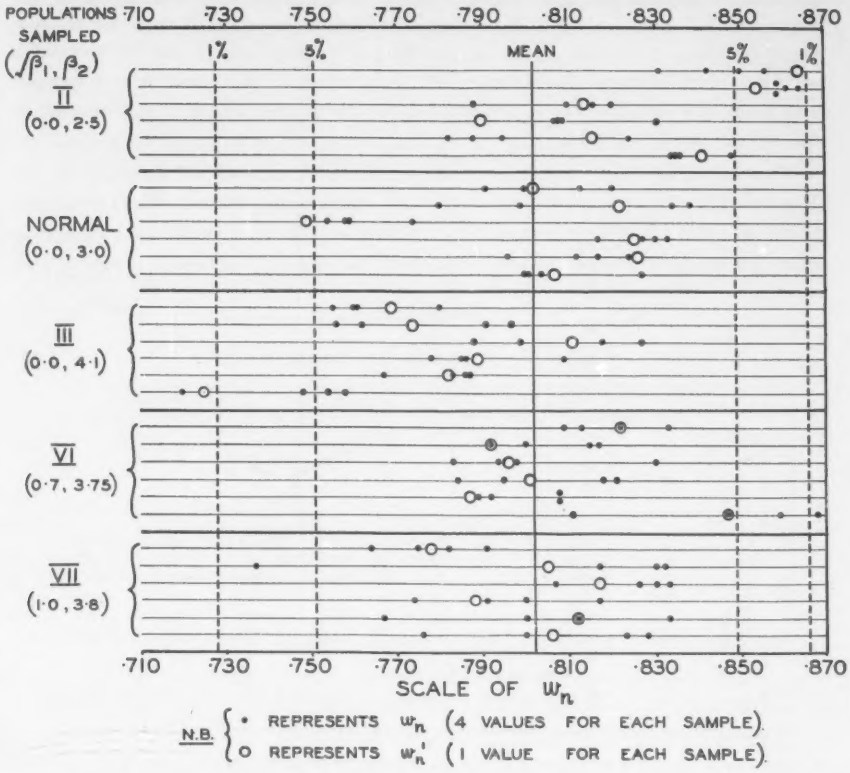
and upper 5 °/₀ and 1 °/₀ probability limits and the mean level have been drawn as follows:

Lower limits: 5 °/₀ ·751, 1 °/₀ ·728.

Upper limits: 5 °/₀ ·849, 1 °/₀ ·866.

Mean level ·802.

FIG. 1. VALUES OF w_n & w_n' IN SAMPLES OF 50.


POPULATIONS SAMPLED ($\sqrt{\beta_1}, \beta_2$)
·710 ·730 ·750 ·770 ·790 ·810 ·830 ·850 ·870
1% 5% MEAN 5% 1%
II (0·0, 2·5)
NORMAL (0·0, 3·0)
III (0·0, 4·1)
VI (0·7, 3·75)
VII (1·0, 3·8)
SCALE OF w_n
N.B. • REPRESENTS w_n (4 VALUES FOR EACH SAMPLE).
O REPRESENTS w_n' (1 VALUE FOR EACH SAMPLE).

The figures are taken from Geary's Table F. The diagram shows clearly that:

(*a*) The four values of w_n vary very considerably about w_n'.

(*b*) w_n' cannot be taken as representative of w_n; it sometimes lies outside the range of the four values.

(*c*) When the population is not Normal very few of the sample points fall beyond even the 5 % limits, so that departures from normality of this order could rarely be detected with samples as small as 50. As far as can be judged from this small experience the chance of detection appears on the whole to be no greater using w_n than w_n'.

The labour involved in the calculation of even a single value of w_n is very considerable, and since for n' as large as 50 there is every reason to believe that w_n' follows closely the same sampling distribution as w_n, it appears that this simpler criterion can be used; in the later experiments therefore w_n' alone has been calculated.

In the case of populations VI and VII the ratio of mean deviation to standard deviation was ·789 and ·797 respectively, differing only very slightly from the Normal value of $\sqrt{2/\pi} = ·798$. It was therefore not to be expected that the w_n' test would be efficient in these cases. In the case of populations II and III the ratios are ·818 and ·768 respectively, but even here owing to the small size of sample, only 2 out of 6 and 1 out of 6, respectively, of the sample values of w_n' are seen to fall beyond the 5 °/₀ limits.

The values of $\sqrt{\beta_1}$ and β_2 were also calculated for each of the six samples; for these criteria of course no change occurs on rearranging the x's. The results are shown in Table III and plotted in Fig. 2, where 5 °/₀ and 1 °/₀ limits for $\sqrt{\beta_1}$ appropriate for samples from a Normal population have been inserted*. The $\sqrt{\beta_1}$ test

TABLE III.

Values of $\sqrt{\beta_1}$ and β_2 in Samples of 50.

Sample No.	Population II		Population III		Population VI		Population VII	
	$\sqrt{\beta_1}$	β_2	$\sqrt{\beta_1}$	β_2	$\sqrt{\beta_1}$	β_2	$\sqrt{\beta_1}$	β_2
1	−·229	1·962	·484	3·753	·124	2·513	1·086	3·666
2	·201	2·122	·718	4·181	·903	3·289	1·131	3·627
3	·215	2·309	−·511	2·517	·856	2·984	·674	2·915
4	·263	2·282	−·294	2·931	·175	2·580	·906	2·934
5	·282	2·332	·434	3·054	·743	3·203	1·022	3·601
6	·161	2·308	·900	5·645	·409	2·301	·994	3·427
Mean	+·149	2·219	+·288	3·680	+·535	2·812	+·969	3·362

is reasonably efficient in detecting the skewness of populations VI and VII; for the other two cases (II and III) where the β_2 test would be appropriate, no limits are available. Since however for $n' = 50$, $\sigma_{\beta_2} = ·60$ approximately, it is doubtful whether significant departures from normality could be established in the case of any single sample even if the true sampling distribution were known. It is of interest to note that in all but 2 of the 24 samples, β_2 is less than the corresponding population value. This bias which appears also in the other experiments will be referred to again below.

Series B.

A second experiment consisted in drawing six samples of 300 from each of the two symmetrical Non-normal populations II and III. In this case the 5 °/₀ and 1 °/₀ limits for β_2 previously calculated are probably fairly accurate, so that a satisfactory comparison of the β_2 and w_n' tests is possible. The results are shown in Table IV, p. 340, and Fig. 3.

* These are obtained from my table already referred to, *Biometrika*, Vol. XXII. (1930—1931), p. 248.

FIG. 2. VALUES IN SAMPLES OF 50 OF :-

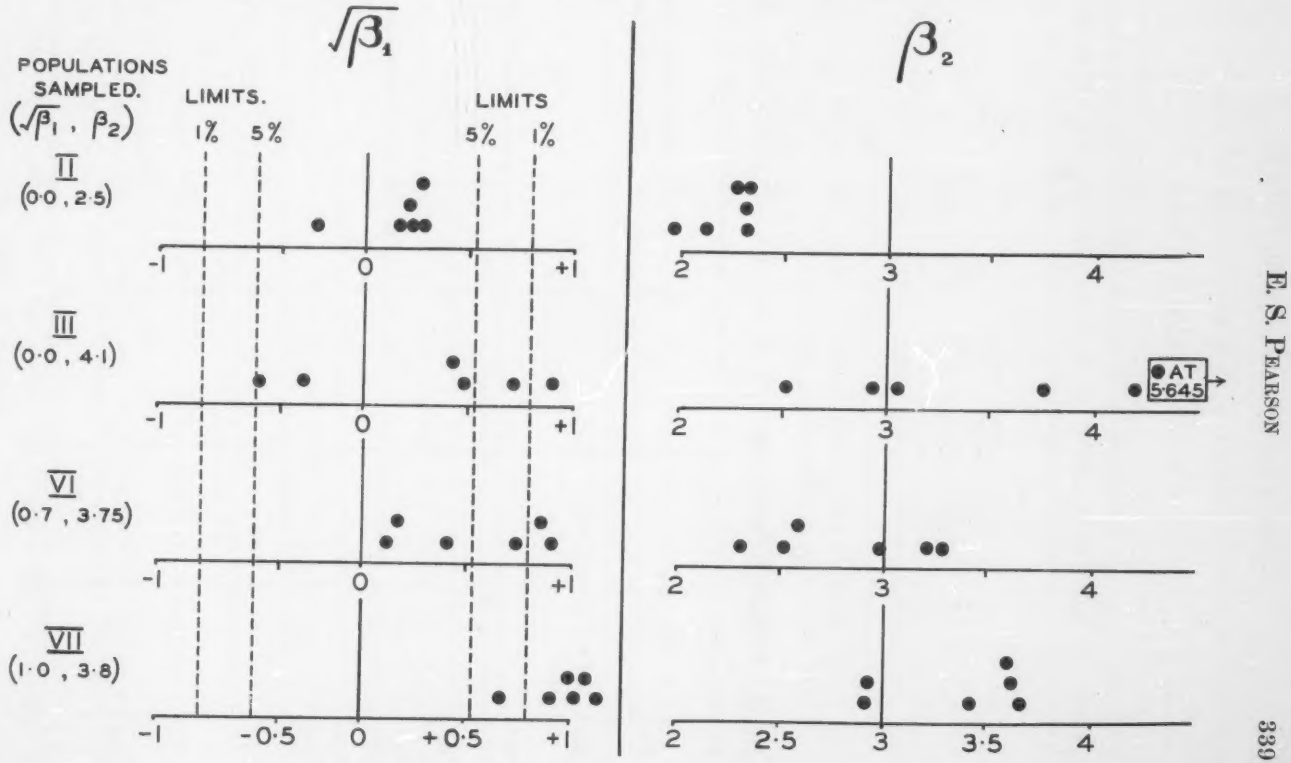
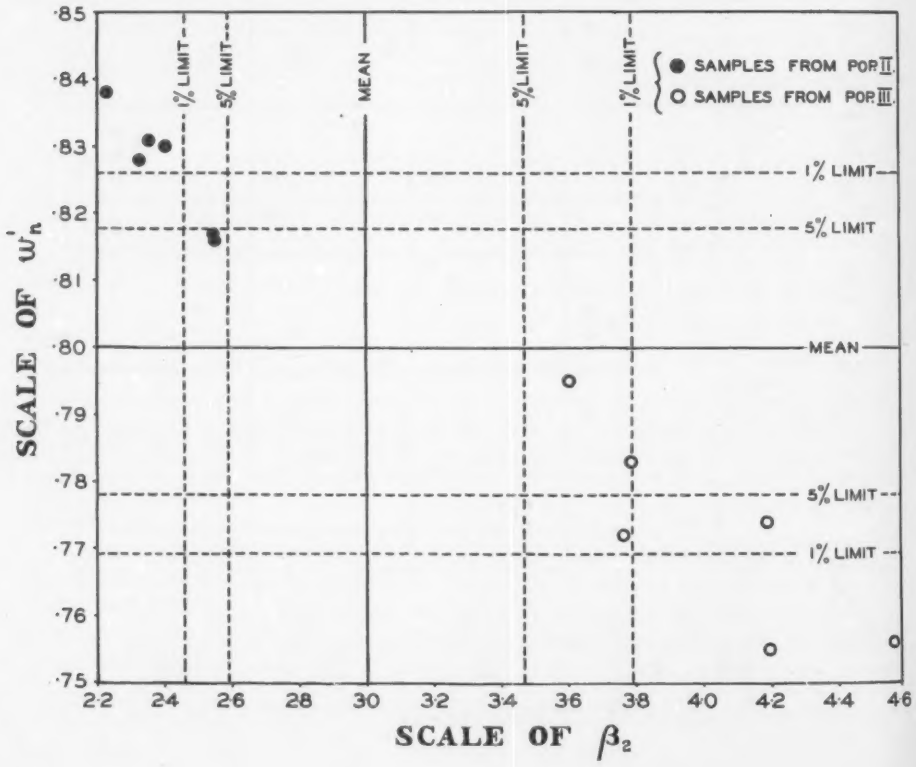


TABLE IV.

Samples of 300; *Values of Criteria.*

Sample No.	Population II (0·0, 2·5)			Population III (0·0, 4·1)		
	$\sqrt{\beta_1}$	β_2	w_n'	$\sqrt{\beta_1}$	β_2	w_n'
1	−·021	2·299	·838	+·490	4·189	·774
2	−·006	2·328	·828	−·300	3·788	·783
3	+·074	2·548	·816	−·020	4·196	·755
4	−·067	2·352	·831	−·198	3·762	·772
5	+·013	2·404	·830	+·249	3·602	·795
6	−·022	2·545	·817	−·456	4·573	·756
Mean	−·005	2·413	·827	−·023	4·018	·772

FIG. 3. COMPARISON OF β_2 & w_n' IN SAMPLES OF 300.



The correlation between the two criteria is extremely high*, so that a decision based on a knowledge of w_n' is not likely to differ much from that based on β_2 and vice versa. There is certainly a suggestion as indicated in the following table that β_2 is somewhat more efficient in detecting lack of normality than w_n', but with only six samples available the evidence is far from conclusive. The mean values of β_2 are slightly less and the mean values of w_n' slightly more than the population values given in Table I, but the bias is far less than in the samples of 50 (Series A) and 76 (Series C).

TABLE V.

Samples of 300; *Comparison of Criteria.*

	Population II		Population III	
	Using β_2	Using w_n'	Using β_2	Using w_n'
Number of sample points:				
Within 5 °/₀ limits	0	2	0	2
Between 5 °/₀ and 1 °/₀ limits	2	0	3	2
Beyond 1 °/₀ limits	4	4	3	2

$\sqrt{\beta_1}$ varies of course about zero, but the standard error increases as the population becomes more leptokurtic† and the values for the 1st and 6th samples for population III (+·490 and −·456) fall far outside the 1°/₀ limits (±·329) for a Normal population. This does not however mean that the $\sqrt{\beta_1}$ test should be used in this case, since the modal value of $\sqrt{\beta_1}$ is zero.

Series C.

In this case ten samples of 76 were drawn from each of the populations I, IV and V; the populations differ more from the Normal than in the preceding case but the samples are considerably smaller. Values of $\sqrt{\beta_1}$, β_2 and w_n' are shown in Table VI, p. 343, and the last two are plotted in the form of a correlation diagram in Fig. 4. In this case no reliable 5°/₀ and 1°/₀ limits for β_2 are available. As a rough indication of their position, however, the procedure adopted in my earlier paper has been followed, i.e. the first four moment coefficients appropriate for $n'=76$ have been calculated‡ and the probability levels obtained by quadrature of the Pearson Type IV curve having these moments. The results were as follows:

Lower limits: 1°/₀ at 2·14, 5°/₀ at 2·27.

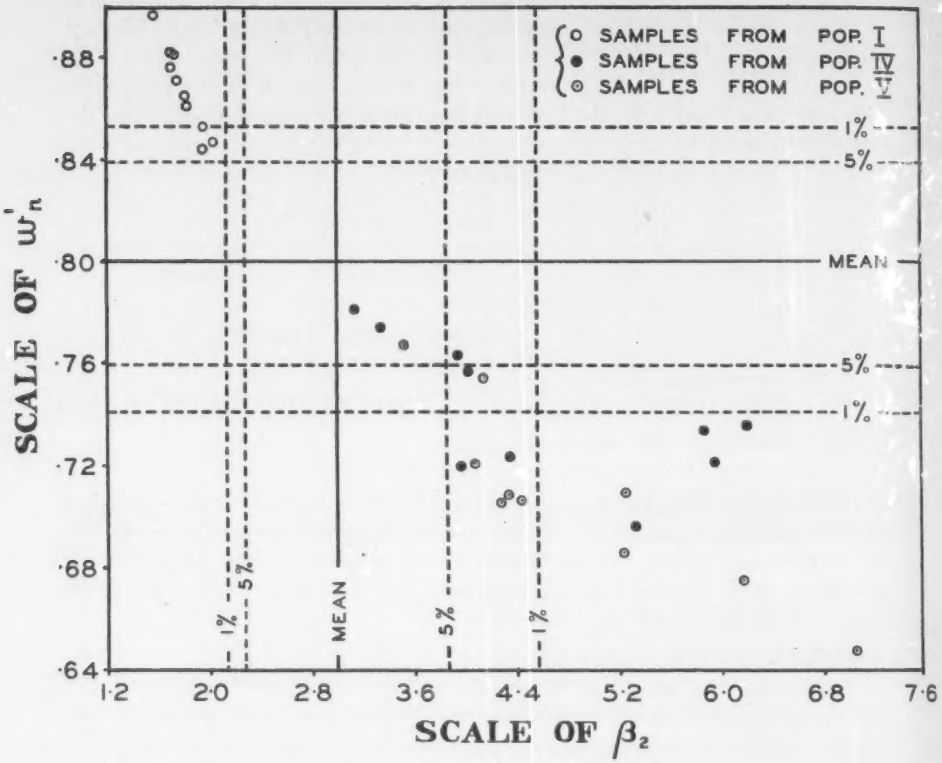
Upper limits: 5°/₀ at 3·86, 1°/₀ at 4·57.

* See Mr Geary's Section III below.

† First approximation values of the standard errors of β_1 and β_2 for samples from different Pearson type curves may be obtained from *Tables for Statisticians and Biometricians*, Part I. Tables XXXVII and XXXVIII.

‡ The formulae used are (5) to (8) of my paper in *Biometrika*, Vol. XXIII. (1931), p. 423, which give mean $\beta_2=2\cdot922$, $\sigma_{\beta_2}=0\cdot510$, $B_1(\beta_2)=1\cdot971$, $B_2(\beta_2)=7\cdot460$. The calculations were kindly undertaken by Mr K. Williams.

FIG. 4. COMPARISON OF β_2 & w_n' IN SAMPLES OF 76.



Turning to the diagram it is seen that there is again a high correlation between the two criteria β_2 and w_n', which is less intense for the leptokurtic populations than for the rectangular. Counts of the points falling beyond different limits leads to Table VII on the opposite page, analogous to Table V. In view of the uncertainty of the position of the β_2 limits, it would not be justifiable to make any critical comparison of relative efficiency of the two tests in this case, but there is certainly no evidence that the w_n' test is less efficient than the β_2 test.

A noticeable feature of Table VI is the bias both in β_2 and w_n' for samples from leptokurtic populations. For the case of normal populations this bias can be corrected by the use of multiplying factors $\frac{n'+1}{n'-1}$ (exactly for β_2) and $e^{-\frac{1}{4(n'-1)}}$ (approximately for w_n'), but these factors will clearly differ for other populations. It will be seen that for Population IV the mean of the 10 sample values of β_2

TABLE VI.

Samples of 76; Values of Criteria.

Sample No.	Population I (0·0, 1·8)			Population IV (0·0, 7·1)			Population V (0·0, 5·9)		
	$\sqrt{\beta_1}$	β_2	w_n'	$\sqrt{\beta_1}$	β_2	w_n'	$\sqrt{\beta_1}$	β_2	w_n'
1	+·090	1·821	·865	+·190	4·038	·757	−·452	5·250	·710
2	+·315	1·839	·861	−1·425	6·200	·736	+·443	6·189	·675
3	+·282	1·709	·876	−·797	5·957	·722	+·822	3·509	·767
4	+·039	1·582	·897	−·755	3·937	·763	+·454	4·441	·707
5	−·232	1·946	·844	+·575	5·874	·734	+·292	5·240	·686
6	+·198	1·709	·882	+·663	3·143	·781	−·169	4·285	·705
7	−·232	1·933	·853	+·570	3·334	·774	−·680	4·093	·721
8	+·315	1·735	·881	+·033	5·323	·697	−·219	4·336	·709
9	+·226	2·044	·847	−·688	4·350	·724	+·106	4·148	·754
10	−·032	1·772	·871	−·442	3·980	·720	−·390	7·149	·648
Mean	+·097	1·809	·868	−·208	4·614	·741	+·021	4·864	·708

TABLE VII.

Samples of 76; Comparison of Criteria.

	Population I		Population IV		Population V	
	Using β_2	Using w_n'	Using β_2	Using w_n'	Using β_2	Using w_n'
Number of sample points:						
Within 5 % limits	0	0	2	3	1	1
Between 5 % and 1 % limits	0	2	4	1	5	1
Beyond 1 % limits	10	8	4	6	4	8

TABLE VIII.

Samples of 76; Comparison of Criteria.

Estimate of standard error of:	Population I	Population IV	Population V	Exact values for Normal population
w_n'	·017	·027	·035	·024
β_2	·137	1·129	1·106	·510
$\sqrt{\beta_1}$	·221	·710	·455	·270

(4·61) falls short of the population value (7·1) by over 33 %*. The bias in w_n', here in the opposite direction, is relatively of much less importance. In the same way, while it will be seen that the standard error of w_n' increases as the population becomes more leptokurtic, this increase is very definitely less than that in the standard error of β_2. Let the reader examine, for example, the sets of 10 samples in the Tables on p. 343. The results given there suggest that in samples of moderate size w_n' provides a better estimate than β_2 of its corresponding population parameter. The problems of estimation and of testing the hypothesis of normality are not however identical, and much more work is necessary before a final comparison of the merits of these two criteria can be made.

3. *Conclusions.*

(1) In testing for departure from normality it would no doubt be desirable to have a single test sensitive both as regards skewness and kurtosis. As this is not available it is necessary to apply two separate tests.

(2) To detect skewness the best criterion would appear to be $\sqrt{\beta_1}$, for which tables of 5 % and 1 % probability levels, believed to be sufficiently accurate for all ordinary purposes are available†.

(3) As a test of whether the population sampled is platykurtic or leptokurtic, either β_2, w_n or w_n' may be used. Out of these there are strong practical grounds for choosing w_n', the ratio of mean deviation to standard deviation, unless the sample is very large when β_2 may be used. The reasons for this are as follows:

(*a*) Until further investigation has been carried out, the accuracy of the tables of 5 % and 1 % probability levels for β_2 cannot be established.

(*b*) While the sampling distribution of w_n is known and accurate tables have been given by Mr Geary, the calculation of this criterion is very troublesome except in the case where the population mean is given.

(*c*) There are strong grounds for believing that if $n' > 40$, w_n' may be referred to the w_n tables of 5 % and 1 % probability levels.

(*d*) The calculation of w_n' is straightforward and shorter than that of β_2; if the data are grouped an approximate adjustment to obtain the mean deviation may be used‡.

* This bias may be explained as follows. The high value of β_2 for the distribution law of equation (1), is due to the very long drawn-out tails. The chance however of obtaining an observation falling in these tails in samples even as large as 76 is small, so that the sample β_2's have a very skew sampling distribution with mode well below the population value. The standard error is however infinite.

† *Tables for Statisticians and Biometricians*, Part II. Table XXXVII *bis* for $n' = 50, 75, 100 \ldots$etc. and *Biometrika*, Vol. XXVII. (1934), pp. 269—271, for $n' = 20, 30, \ldots 90, 100$.

‡ If the centre of the group containing the mean, $\bar{x}$, of the sample be taken as the origin of x, if n_t observations fall into the tth group having central value x_t, if n_0 observations fall into the group containing the mean and if the mean is at a distance θ from the origin $(-\frac{1}{2} \leqslant \theta \leqslant \frac{1}{2})$ where the unit is the group breadth, then we may take as an approximation

$$\Sigma n_t |x_t - \bar{x}| = \Sigma n_t |x_t| + \{\theta \times \text{number of observations in groups below the origin group}\} - \{\theta \times \text{number of observations in groups above the origin group}\} + n_0(\tfrac{1}{4} + \theta^2).$$

(*e*) There is clear experimental evidence that in the case of leptokurtic populations the sample β_2 provides a biased estimate of the population value. This bias is far smaller in the case of w_n'.

(4) A high negative correlation exists between the sample values of β_2 and w_n'*. This means that in testing the hypothesis of normality, in any given case a conclusion drawn from the value of one criterion is unlikely to differ much from that which would be drawn from a knowledge of the other. If the uncertainty regarding the probability levels of β_2 could be removed, it is likely that for large samples this criterion would be found somewhat more efficient than w_n' in detecting departure from normality. This point cannot however be regarded at present as established.

* See Section III added by Mr Geary giving the value for this correlation when the population sampled is Normal.

[NOTE. The object of publishing the following Appendix containing 72 random samples of various sizes from diverse forms of curves is to put into the hands of those desiring to test other, old or new, criteria ready-made samples and save them from the heavy labour of obtaining afresh their own samples. ED.]

APPENDIX.

FULL DATA CONCERNING POPULATIONS AND SAMPLES USED IN A PAPER BY E. S. PEARSON ENTITLED: "A COMPARISON OF β_2 AND MR GEARY'S ν_n CRITERIA."

For the purpose of future reference and comparison, it has appeared useful to publish in this appendix full details of the populations sampled and the method of sampling, and also to give the complete data of the samples themselves.

In sampling with Tippett's Random Numbers (*Tracts for Computers*, No. XV) the method of procedure described on pp. iv—v of the introduction to the Tract has been employed. The frequency distributions of the 10,000 numbers representing each population are shown in Table X, except in the case of Population I (Rectangular); here the sampling was arranged so that there was an equal chance (i.e. $\frac{1}{33}$) of the variable, x, assuming values $-16, \ldots -1, 0, +1, \ldots +16$. This was effected by assigning the numbers 01—03 to $x = -16$, 04—06 to $x = -15$, ... 97—99 to $x = +16$, omitting the number 00 from consideration.

In apportioning the 10,000 numbers it is inevitable that difficulty should arise in the case of very leptokurtic distributions. This difficulty was discussed in a note contributed to Dr le Roux's paper, to which the reader is referred*. It means that the actual populations sampled cannot correspond exactly to those defined by the theoretical curves, but from the point of view of the present investigation this is of no consequence, since the objective has been to investigate the efficiency of various criteria in detecting certain wide departures from normality.

The frequency constants shown in Table IX have been calculated from the actual distributions sampled, i.e. from those given in Table X, without applying

TABLE IX.

Frequency Constants of Sampled Populations.

	I	II	Normal	III	IV	V	VI	VII
Mean	0·000	0·000	0·000	0·000	0·000	0·000	0·142	0·147
Standard Deviation	9·522	10·544	10·004	5·675	6·455	3·546	5·008	12·370
Mean Deviation	8·250	8·629	7·986	4·378	4·757	2·483	3·961	9·860
$\sqrt{\beta_1}$	0·000	0·000	0·000	0·000	0·000	0·000	0·701	0·995
β_2	1·798	2·500	3·000	4·116	7·052	5·884	3·725	3·834

* *Biometrika*, Vol. XXIII. pp. 158—159.

any corrections for grouping or abruptness except in the case of the mean deviation where the expression given in the footnote ‡ on p. 344 above was used. The unit employed in the case of the mean, standard deviation and mean deviation is the unit of x shown on the x-scales of Table X. It will be seen that this varies from about $\frac{2}{7}$ to $\frac{1}{12}$ of the standard deviation; it would, no doubt, have been desirable to arrange a finer grouping unit in the case of Populations III—VI, but the data were collected from sampling experiments initiated at different times and with different objectives.

The sampling results are shown in Tables XI—XIII; for Experiment A, the 50 individual values of x are recorded for each sample. These values are tabled in a completely random order; in fact the order is that used in calculating the first of the 4 sets of values of w_n given in Table II. The rearrangements needed to obtain the other 3 values of w_n were carried out by writing the 50 x-values on small counters, thoroughly mixing and drawing out randomly in order.

For Experiments B and C the tables show the frequency of occurrence in each sample of the different values of x. Thus in Table XII the columns each total 300 and in Table XIII they total 76.

The sample frequency constants tabled in the preceding paper have been calculated directly from these figures without further grouping. No Sheppard's Corrections have been applied, either to the second or fourth moments; the mean deviation was calculated from the expression given on p. 344 above*.

* Further reflection suggests that if Sheppard's Corrections are not used, the correction to the mean deviation to adjust the contribution from the central group is not really justifiable. The consistent course would, no doubt, have been to suppose that x could assume certain discrete values only; the resulting alterations to the values of the mean deviation would, however, be too small to be of practical significance.

TABLE X.

Grouped Frequencies in Populations Sampled.

x	Population frequencies for symmetrical distributions					Population frequencies for skew distributions					
						VI, $\sqrt{\beta_1}=0{\cdot}7$, $\beta_2=3{\cdot}75$		VII, $\sqrt{\beta_1}=1{\cdot}0$, $\beta_2=3{\cdot}8$			
	II $\beta_2=2{\cdot}5$	Normal	III $\beta_2=4{\cdot}1$	IV $\beta_2=7{\cdot}1$	V $\beta_2=5{\cdot}9$	x	Frequency	x	Frequency	x	Frequency
0	348	398	774	758	1812	−12	1	−17	141	+23	62
± 1	348	397	758	740	1350	−11	9	−16	245	+24	57
± 2	345	391	710	690	905	−10	37	−15	298	+25	52
± 3	340	381	637	615	606	− 9	98	−14	332	+26	47
± 4	332	368	550	529	406	− 8	197	−13	353	+27	43
± 5	322	352	459	440	273	− 7	327	−12	367	+28	39
± 6	311	334	369	355	183	− 6	472	−11	375	+29	35
± 7	298	312	291	282	122	− 5	611	−10	377	+30	32
± 8	284	289	222	221	82	− 4	726	− 9	376	+31	28
± 9	268	266	168	170	55	− 3	802	− 8	373	+32	26
±10	251	242	123	130	37	− 2	839	− 7	366	+33	23
±11	234	218	91	100	25	− 1	834	− 6	358	+34	21
±12	214	195	66	77	16	0	796	− 5	349	+35	18
±13	196	171	47	58	11	+ 1	734	− 4	339	+36	16
±14	177	150	34	45	8	+ 2	654	− 3	327	+37	15
±15	159	129	24	35	5	+ 3	569	− 2	315	+38	13
±16	139	111	18	27	3	+ 4	482	− 1	303	+39	11
±17	122	94	12	21	2	+ 5	400	0	290	+40	9
±18	104	79	9	16	2	+ 6	325	+ 1	277	+41	9
±19	88	66	7	13	1	+ 7	260	+ 2	264	+42	8
±20	74	54	5	10	1	+ 8	205	+ 3	252	+43	6
±21	60	44	3	8	—	+ 9	159	+ 4	239	+44	6
±22	46	36	3	7	—	+10	121	+ 5	226	+45	5
±23	37	28	1	5	1	+11	92	+ 6	213	+46	4
±24	27	23	2	5	—	+12	69	+ 7	202	+47	3
±25	19	17	1	3	—	+13	51	+ 8	190	+48	3
±26	14	14	1	3	—	+14	37	+ 9	178	+49	3
±27	8	10	—	2	—	+15	27	+10	168	+50	2
±28	5	8	1	2	—	+16	20	+11	157	+51	2
±29	3	6	—	2	—	+17	14	+12	146	+52	1
±30	1	5	—	2	—	+18	9	+13	137	+53	1
±31	—	3	—	1	—	+19	7	+14	128	+54	1
±32	—	2	—	1	—	+20	5	+15	119	+55	1
±33	—	2	—	—	—	+21	4	+16	110	+56	1
±34	—	1	1	1	—	+22	2	+17	102	+57	—
±35	—	1	—	1	—	+23	2	+18	94	+58	1
±36	—	1	—	—	—	+24	1	+19	87	+59	—
±37	—	—	—	1	—	+25	—	+20	80	+60	—
±38	—	—	—	1 at ±40	—	+26	1	+21	74	+61	—
±39	—	1	—	1 at ±47	—	+27	—	+22	68	+62	1
±40	—	—	—	1 at ±58	—	+28	1	continued above			
Totals	10,000	10,000	10,000	10,000	10,000	Total	10 000			Total	10,000

N.B. For Population I (Rectangular) see special reference in text.

TABLE XI.

Experiment A. Individual Values of x in Six Samples of 50 from Five Populations.

Population II						Normal Population						Population III		
+ 9	+ 5	+ 1	0	+12	+ 5	+ 5	0	− 3	+10	+15	− 6	+ 2	− 6	+ 1
+ 1	+ 4	0	− 9	+11	−19	−10	− 4	+16	− 3	+12	+ 3	+ 4	− 1	− 5
+ 5	− 7	+ 8	− 2	−12	+16	− 8	+ 9	− 3	+15	+ 6	− 4	+ 4	+ 4	0
− 6	+12	−21	+21	+ 7	+10	− 8	− 4	−16	+11	+17	+10	− 8	− 2	− 1
+ 5	+22	+17	−15	+11	−13	− 8	+15	−28	+ 4	0	+ 4	+ 3	+ 1	− 2
+15	+18	−12	−15	− 8	− 7	+ 9	+ 5	+15	+16	+ 3	+ 3	+11	− 3	− 7
−15	+10	+26	+15	− 14	−19	+22	0	− 7	+13	+ 9	− 5	+ 4	− 5	− 5
−13	− 6	−13	−22	− 3	− 5	− 2	−16	+ 2	− 1	+ 6	+ 1	0	0	+ 4
+12	+ 1	+10	+ 2	+ 7	− 9	+12	0	+ 1	− 9	+ 2	− 3	− 6	+ 3	− 1
−18	−10	+ 6	− 6	+ 5	+ 9	+ 8	− 6	− 3	+ 1	+ 7	−10	− 4	+ 6	+ 6
+13	− 7	+ 1	− 5	+ 2	+ 4	− 5	− 9	0	+ 1	+ 3	0	− 3	+ 4	− 1
+19	+ 6	−22	+ 1	−13	0	+ 4	− 2	0	+ 4	+11	+10	− 3	+ 9	− 5
− 3	− 1	+19	+13	− 3	− 3	− 7	+16	+18	−15	−11	+ 1	+ 7	+ 7	− 2
+10	−22	+13	+ 2	− 1	+ 5	0	+11	− 5	+12	+ 9	− 6	− 3	+ 2	−10
+15	+16	− 9	−19	−15	+ 4	+14	− 4	+ 2	−18	− 9	−13	− 7	+ 4	+ 3
+ 1	−10	0	− 3	−14	+ 4	− 3	−17	− 2	+10	− 6	− 5	+ 3	+10	− 2
−15	+ 6	+ 2	− 8	+20	+ 9	− 5	− 4	− 6	+21	+14	− 7	− 1	− 4	− 1
− 5	−23	−16	− 4	− 1	+23	−12	+ 2	+ 3	+ 9	− 1	−12	− 8	−13	− 1
+ 2	+ 9	− 4	+15	− 13	−10	−12	+ 5	−13	−10	− 2	−12	+ 6	− 8	− 1
+16	+ 3	− 2	+ 9	0	− 9	+13	− 4	−21	−14	+15	+ 8	− 2	+ 9	+ 5
+15	−10	+ 2	0	+15	−14	−27	− 3	−14	− 7	−12	+11	− 2	− 7	+ 6
+10	− 3	− 2	− 5	+15	− 1	− 7	+13	+12	+ 5	+ 1	− 4	+ 2	− 4	− 9
−11	−11	−19	+ 5	− 3	−17	−10	− 5	+ 3	+ 2	− 3	− 9	+ 3	+12	0
+ 4	− 1	+17	+ 3	0	−12	+ 8	− 6	− 1	−10	− 8	+ 4	− 5	+ 2	−12
+ 1	− 7	+ 1	−19	+20	−10	0	− 2	+ 4	− 7	− 5	+ 1	+ 1	− 2	− 8
−13	−11	− 8	− 2	0	−16	− 7	− 2	+ 8	− 3	− 9	−11	− 8	+ 3	− 6
−18	− 6	−14	− 7	−11	− 8	− 2	+14	+ 6	−12	− 2	−13	− 7	− 1	+ 6
+ 7	+11	− 7	+11	+26	0	−13	+ 1	− 2	− 7	− 3	+ 3	+15	− 1	0
−10	− 4	+ 8	− 6	− 9	− 6	+ 2	+ 5	+ 8	−22	+ 2	+ 9	− 2	− 1	−10
− 8	−22	−23	+18	+16	+14	+ 1	− 7	+ 1	− 4	0	− 1	− 1	+ 1	+ 7
−18	− 2	− 9	−14	− 6	− 9	− 4	− 6	− 5	+ 1	−16	− 3	− 2	− 2	0
−25	+20	−11	+13	+ 3	− 3	−12	− 5	+19	+ 9	+12	+ 3	− 1	− 8	+ 4
+13	+ 7	−11	− 4	+11	+ 9	+20	+22	− 2	− 4	− 5	+14	0	− 6	+ 2
− 7	−17	−10	− 4	+ 2	− 8	+ 3	0	+ 3	− 2	+ 1	+ 5	+ 3	− 2	+ 2
+19	− 3	− 6	−13	+19	+ 3	+14	+ 7	+ 8	+17	+11	+15	− 7	−11	+ 4
−18	−12	+ 6	+14	− 18	+ 2	+10	+ 6	−12	+ 3	+ 3	− 3	− 4	+ 1	+ 4
+10	− 8	0	+ 3	+ 3	+11	+ 7	+ 9	−24	− 4	+10	0	− 1	+ 8	− 4
+ 6	−12	+21	− 2	− 2	+10	0	− 3	+ 6	−12	−14	− 2	− 1	− 6	+ 3
+17	− 5	−19	−14	− 2	+ 9	− 2	− 1	− 1	− 1	−21	− 2	− 1	− 7	+ 3
− 4	+13	− 2	−14	− 4	+ 2	−13	+21	+ 1	+ 7	−17	+20	− 1	− 8	+ 4
+ 6	−14	+11	− 8	− 2	− 7	+ 8	+10	+15	+ 9	+15	− 1	+ 3	− 3	− 2
+ 9	−12	+16	−12	− 4	+16	+ 7	+ 3	− 2	+ 7	+10	− 4	+ 3	−11	− 5
+11	−13	+ 2	− 5	− 6	−12	− 8	− 6	− 7	−11	− 9	+17	+ 2	+ 2	+ 2
−19	+ 7	+19	−18	+10	− 7	+ 9	+15	+ 6	0	− 2	+ 3	−11	+23	+ 3
− 8	−13	+ 2	− 3	+12	+ 4	− 7	− 8	−15	− 8	− 6	− 5	+ 1	+ 8	− 6
− 2	−16	−13	0	+14	− 1	+30	+ 5	− 2	+ 2	+ 8	− 9	− 3	− 3	+ 2
− 3	+13	− 9	−23	− 2	− 6	+ 2	−11	+20	− 8	− 9	+ 3	− 2	− 5	+ 4
− 3	+11	− 7	+20	−13	+15	−10	+14	−16	+ 8	−10	−14	+ 3	− 4	− 1
− 8	−10	− 3	+16	− 2	0	− 7	−15	+11	− 8	− 5	− 9	+ 9	− 8	− 3
+ 6	+13	+14	−19	− 1	+ 5	+ 3	+ 7	− 1	+ 1	+25	+ 9	− 1	−14	+ 4

TABLE XI (*continued*).

Population III			Population VI						Population VII					
+ 5	− 1	+ 1	− 6	− 1	− 4	+ 4	+ 1	− 2	− 6	−12	0	− 5	+27	− 8
0	+ 4	+23	− 4	+ 3	+10	+ 9	0	− 2	0	− 5	−12	−11	+ 1	+35
+ 1	− 3	− 2	− 6	0	− 5	− 6	− 5	+ 7	−17	+21	− 6	− 3	− 8	+ 5
−12	+ 1	− 7	+ 5	− 3	+ 7	− 2	− 2	+ 1	− 9	+11	+20	+ 4	+10	− 3
+ 4	+ 6	+ 3	− 6	+ 3	+ 9	+ 2	0	− 2	+11	−15	−17	− 9	− 7	−11
+ 1	− 6	− 1	− 2	− 6	− 3	+11	− 6	− 1	−14	− 5	+ 9	− 8	+ 9	− 3
− 5	+ 1	− 2	+ 3	− 7	− 5	− 2	+ 2	− 2	−11	− 4	−12	+ 1	+ 2	−15
− 5	0	0	−11	+ 1	+11	0	+ 5	+ 1	+11	+36	+ 4	−10	+17	+ 9
+ 5	− 3	+ 3	+ 3	− 6	+ 3	− 5	− 7	+ 2	− 2	−15	− 4	+20	+ 7	+33
− 3	+ 1	+ 4	+ 1	+10	+15	− 3	− 5	0	− 8	+34	+ 3	−12	− 8	+ 4
− 4	+ 7	+11	− 2	− 5	+ 3	− 1	− 1	0	+19	+19	+19	+ 6	−11	− 9
+ 4	− 1	− 2	+ 7	− 1	+ 8	− 1	− 8	− 3	+ 2	− 4	− 4	− 9	− 1	+ 4
+ 3	+14	+ 4	− 1	− 3	− 2	− 4	0	− 5	+15	−15	−16	0	− 6	+ 9
+ 1	− 7	+ 7	−10	− 8	− 3	0	− 6	+10	−13	−12	−16	− 1	+ 9	−10
− 3	+ 4	+ 8	− 2	− 3	− 7	− 2	+14	− 1	−13	+ 7	−10	+18	+12	− 6
+ 7	− 1	− 4	− 3	+12	− 5	− 4	+ 3	− 4	− 5	+ 5	−13	+ 5	− 4	+ 1
− 1	− 4	− 7	− 8	− 1	− 2	+ 3	0	+11	−12	+22	− 9	− 1	− 5	+ 2
+ 1	− 1	− 5	− 1	− 4	0	− 1	+ 7	0	0	−11	− 9	− 4	− 7	−12
0	− 4	+ 2	+ 1	− 4	− 4	+ 3	− 2	− 4	−13	− 6	−10	+ 2	+ 6	− 9
− 5	− 4	+ 4	+ 1	− 5	− 2	+ 4	− 6	− 4	−11	+12	−16	− 7	+ 6	+ 9
0	− 9	+ 2	+ 3	0	− 5	+ 6	− 6	+ 7	−15	− 9	−15	+ 3	+19	+16
+ 2	+ 5	+ 2	− 5	− 1	− 2	+ 3	− 5	+ 7	− 1	−10	− 1	− 9	0	−12
− 2	+ 5	+ 1	+ 2	+10	− 3	+ 6	− 1	+ 7	−10	− 9	0	0	− 6	+11
+ 7	+ 2	+ 4	+ 3	+ 3	− 6	0	− 4	− 9	−14	− 4	−11	− 8	− 3	+ 6
+ 1	− 2	+ 1	+ 1	− 4	− 6	0	+ 4	0	0	− 2	+19	− 3	−11	− 2
−11	+ 1	− 8	+ 4	+ 3	0	+ 4	+ 3	+13	+ 4	− 2	− 2	+10	+ 1	+30
+ 3	− 7	+ 6	+ 3	+ 3	+ 2	+ 3	− 3	+ 5	−16	−12	+ 5	+ 1	+25	− 8
− 3	0	− 6	+ 8	− 5	− 2	− 6	+ 2	− 2	− 3	+16	+ 3	+ 5	−11	+20
− 5	− 1	+ 1	+ 6	− 4	− 4	+10	+11	+ 8	−11	− 9	+ 5	+18	− 9	−16
− 1	− 9	+ 6	− 1	− 4	− 4	− 8	+ 1	− 3	−15	− 7	− 1	+19	+ 7	− 8
+ 5	− 4	− 1	+ 9	0	− 5	+10	+ 6	− 5	− 9	− 6	− 3	0	− 8	−14
+ 4	+12	+13	0	+ 2	0	− 2	0	− 1	− 2	− 6	+ 3	+ 4	−13	−16
+ 9	+ 2	+ 4	− 5	− 6	+ 6	+ 2	+ 2	− 3	+14	+ 9	+ 3	− 8	+ 9	−15
+ 2	+ 8	− 3	+ 1	− 3	− 2	− 2	− 1	+ 2	− 7	+10	+27	+27	−15	− 1
− 3	+ 4	0	+ 2	− 5	+ 2	+ 6	− 1	+ 4	− 4	−13	− 6	− 7	+19	−17
− 1	+ 1	+ 1	− 7	0	+ 2	+ 4	+13	− 7	− 7	−11	−17	− 4	+25	+ 7
+ 3	+ 7	− 1	+ 5	− 7	+ 4	− 1	+ 4	+11	+ 9	0	+10	− 6	− 7	+ 3
− 2	+ 3	+ 1	− 6	+ 2	0	+ 3	− 6	+ 6	+ 1	− 6	+ 9	− 4	+ 3	− 6
0	0	− 2	0	+ 5	+ 1	− 7	− 6	− 3	− 4	− 8	+ 6	+18	+39	+ 6
− 3	+ 7	− 2	− 3	+13	− 8	+ 3	+ 5	+15	− 2	− 3	− 3	− 3	− 5	−13
− 3	+15	−12	+ 8	− 7	− 7	+ 3	+ 4	− 5	+ 5	−11	−17	− 6	− 3	− 6
+ 6	+ 8	− 2	− 4	− 3	+ 1	− 1	+ 2	− 5	− 8	− 7	− 6	− 7	+ 3	−13
+ 3	+ 1	+10	+ 7	− 8	+11	− 8	− 5	+ 1	0	− 1	+ 6	−12	−10	−12
−11	+ 2	+ 5	+12	0	− 2	0	+ 3	+ 5	+20	−13	+ 8	−12	−12	+ 4
0	− 3	+ 4	− 1	+ 3	+ 3	− 9	0	− 9	−11	+ 3	−13	−14	− 7	−14
− 6	− 5	+ 4	− 3	− 3	− 6	+ 1	− 3	+ 8	+29	+14	−16	+16	− 9	− 5
−10	0	− 7	+ 5	0	− 6	− 3	− 3	+10	+ 6	−16	−16	−16	−11	+ 5
− 2	+ 6	+ 2	− 3	− 6	+ 9	− 6	+ 9	+ 7	+29	− 8	− 4	− 9	+11	+14
− 6	+ 6	− 2	− 3	+ 4	0	− 4	− 6	+ 5	− 8	+17	−16	−11	+ 7	− 7
+10	− 3	− 3	− 7	+10	0	+ 5	− 4	− 4	− 6	+ 6	−10	+20	− 8	+30

TABLE XII.

Experiment B. Frequency of Different Values of x in Samples of 300.

x	6 Samples from Population II						6 Samples from Population III						x
−28	—	—	—	—	—	—	—	—	—	—	—	—	−28
−27	—	—	1	1	1	—	—	—	—	—	—	—	−27
−26	2	—	1	1	—	—	—	—	—	—	—	1	−26
−25	2	1	—	—	—	—	—	—	—	—	—	—	−25
−24	2	—	—	—	2	—	—	—	—	—	—	—	−24
−23	—	3	—	1	1	—	—	—	—	—	—	—	−23
−22	1	1	1	2	1	2	—	1	—	—	—	—	−22
−21	3	2	2	2	4	1	—	—	—	—	—	—	−21
−20	1	3	2	3	4	2	—	—	—	—	—	1	−20
−19	2	6	2	6	—	1	—	—	1	1	—	1	−19
−18	1	3	1	4	3	6	—	—	—	1	—	1	−18
−17	3	2	4	4	4	4	1	—	—	1	—	—	−17
−16	10	2	3	3	9	3	1	1	—	—	—	—	−16
−15	7	5	6	4	6	3	—	1	—	—	1	2	−15
−14	8	5	4	5	3	5	—	1	3	2	1	1	−14
−13	7	12	5	6	8	5	—	—	—	1	—	—	−13
−12	3	3	5	5	8	5	2	3	2	3	—	5	−12
−11	7	7	5	12	9	7	—	1	4	—	—	4	−11
−10	11	6	9	4	6	4	8	3	3	6	2	2	−10
− 9	10	5	11	7	9	4	12	6	1	7	7	7	− 9
− 8	6	19	7	7	12	12	4	7	—	7	4	8	− 8
− 7	9	7	7	13	11	5	8	12	11	8	11	4	− 7
− 6	7	11	14	9	13	10	14	9	8	14	11	12	− 6
− 5	11	5	9	7	11	12	10	10	13	12	9	14	− 5
− 4	9	8	13	8	7	12	19	15	14	10	19	16	− 4
− 3	7	17	10	12	8	15	17	18	17	14	19	15	− 3
− 2	6	7	13	7	10	13	15	19	27	14	24	15	− 2
− 1	9	10	11	6	8	7	29	23	23	24	24	17	− 1
0	9	11	6	11	10	10	18	28	21	28	21	29	0
+ 1	8	15	14	7	15	8	16	23	30	28	21	30	+ 1
+ 2	6	4	8	9	6	8	25	16	31	15	17	18	+ 2
+ 3	12	7	8	12	8	18	16	16	24	15	32	19	+ 3
+ 4	9	8	12	11	11	14	18	22	10	22	15	16	+ 4
+ 5	8	8	11	13	15	11	17	18	20	23	14	19	+ 5
+ 6	8	14	10	9	8	10	14	16	7	13	13	14	+ 6
+ 7	11	2	8	6	10	8	10	8	5	7	11	8	+ 7
+ 8	6	6	10	13	5	8	3	11	8	8	8	4	+ 8
+ 9	10	11	7	7	5	8	2	3	2	6	6	4	+ 9
+10	12	9	7	5	8	10	3	1	5	3	2	7	+10
+11	10	8	2	6	12	6	6	5	4	1	2	1	+11
+12	6	8	6	9	5	4	5	2	1	1	2	1	+12
+13	3	6	4	6	1	8	1	—	—	2	—	—	+13
+14	6	6	6	5	3	4	—	—	3	—	1	1	+14
+15	3	7	6	4	4	6	—	—	—	1	—	—	+15
+16	3	7	4	8	5	7	—	—	1	—	1	1	+16
+17	7	3	6	7	3	3	2	—	—	—	1	—	+17
+18	4	2	6	2	2	4	—	—	—	1	—	1	+18
+19	6	1	4	2	1	1	2	1	1	—	—	1	+19
+20	1	2	2	3	2	1	—	—	—	1	1	—	+20
+21	2	1	1	3	1	3	—	—	—	—	—	—	+21
+22	3	2	1	—	1	—	—	—	—	—	—	—	+22
+23	1	—	2	1	1	—	1	—	—	—	—	—	+23
+24	—	—	1	—	—	—	1	—	—	—	—	—	+24
+25	—	1	1	—	—	—	—	—	—	—	—	—	+25
+26	1	1	1	1	—	1	—	—	—	—	—	—	+26
+27	—	—	—	1	—	1	—	—	—	—	—	—	+27
+28	1	—	—	—	—	—	—	—	—	—	—	—	+28

TABLE XIII.

Experiment C. Frequency of Different Values of x in Samples of 76.

x	10 Samples from Population I										10 Samples from Population IV										10 Samples from Population V										x
−30	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	−30
−28	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	−28
**	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	**
−25	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	−25
−24	—	—	—	—	—	—	—	—	—	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	−24
−23	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	−23
−22	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	−22
−21	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	−21
−20	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	−20
−19	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	−19
−18	—	—	—	—	—	—	—	—	—	—	1	—	1	1	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	−18
−17	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	−17
−16	2	1	—	—	1	3	2	2	3	2	1	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	−16
−15	2	—	2	1	2	3	—	4	—	2	—	—	—	1	—	—	1	1	1	—	—	—	—	—	—	—	—	—	—	—	−15
−14	2	4	3	2	4	3	2	4	1	1	1	—	1	1	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	1	−14
−13	4	1	5	4	—	2	5	5	2	—	1	—	—	—	1	—	—	1	1	1	1	—	—	—	1	—	—	—	—	—	−13
−12	2	3	5	3	1	2	3	6	4	3	—	1	—	1	—	—	2	—	1	—	—	—	—	1	1	—	1	—	—	—	−12
−11	3	4	2	3	3	4	4	2	5	2	—	1	1	1	—	—	—	—	1	—	—	1	—	—	—	—	—	1	—	—	−11
−10	3	1	3	2	2	1	2	4	2	3	1	—	—	2	3	—	1	2	—	—	—	—	—	—	—	—	2	—	—	—	−10
− 9	1	7	3	3	2	3	1	—	3	3	1	2	2	2	—	1	1	1	—	2	—	1	—	—	—	—	1	1	1	—	− 9
− 8	4	2	4	4	2	5	—	1	1	5	1	1	2	—	1	2	2	2	2	1	—	—	—	—	—	1	2	1	—	1	− 8
− 7	3	4	1	2	2	5	—	4	2	1	1	4	4	1	—	1	3	2	1	2	2	3	—	1	1	1	1	—	—	1	− 7
− 6	1	—	4	2	—	1	2	4	4	3	3	3	1	2	2	3	1	—	—	—	1	1	—	2	—	2	1	—	—	1	− 6
− 5	2	3	3	1	2	2	2	1	4	2	5	3	3	5	1	4	—	7	—	2	2	—	2	3	3	1	5	2	5	1	− 5
− 4	1	3	1	1	4	2	2	2	1	2	3	8	3	1	6	2	6	1	6	2	—	1	2	3	6	1	2	3	2	5	− 4
− 3	5	6	2	4	1	3	4	3	3	2	4	3	3	7	6	8	2	6	3	2	6	2	6	—	3	5	1	4	4	7	− 3
− 2	2	2	2	4	—	—	3	2	3	2	3	2	7	2	2	4	6	5	6	6	5	7	6	7	5	6	4	3	6	9	− 2
− 1	1	2	2	1	1	3	1	—	3	2	8	10	2	4	7	4	4	4	3	7	7	8	12	11	8	12	6	10	12	9	− 1
0	3	4	2	1	3	1	3	4	3	4	3	6	3	7	5	7	9	6	11	9	16	20	13	10	13	20	21	15	12	14	0
+ 1	3	2	1	2	5	2	2	—	2	1	5	4	9	3	4	7	1	10	5	8	13	9	12	12	10	9	12	15	11	13	+ 1
+ 2	2	—	2	3	1	2	5	1	1	4	6	6	9	6	2	8	4	3	14	8	6	4	3	8	10	9	5	4	9	6	+ 2
+ 3	2	—	5	2	3	1	2	3	4	2	6	5	6	8	7	3	8	3	4	3	5	6	5	4	4	1	4	3	4	3	+ 3
+ 4	3	1	—	1	5	3	5	1	3	1	7	5	1	2	4	1	3	4	4	1	3	6	6	3	2	5	3	4	5	2	+ 4
+ 5	1	2	1	—	4	2	5	1	1	3	2	3	2	6	4	3	1	4	3	1	3	1	2	2	2	1	3	4	2	1	+ 5
+ 6	5	3	1	—	2	—	1	2	4	1	3	2	4	2	6	2	5	2	2	5	3	2	2	2	1	1	2	1	—	—	+ 6
+ 7	2	3	1	5	2	2	7	4	2	2	3	3	2	2	4	5	3	2	1	3	1	1	—	1	—	—	—	—	1	1	+ 7
+ 8	1	1	4	2	3	2	2	2	2	—	3	1	5	1	3	2	1	1	2	2	1	1	2	1	—	1	—	2	—	—	+ 8
+ 9	1	2	1	2	4	2	—	2	2	5	—	—	—	1	1	1	2	1	1	3	—	—	2	—	1	—	—	2	—	—	+ 9
+10	3	1	1	1	2	4	2	1	1	4	1	—	1	4	1	3	2	1	1	3	1	—	1	2	1	—	1	—	1	—	+10
+11	2	—	2	2	1	2	3	1	2	3	—	—	1	1	—	—	1	1	1	—	—	1	—	1	—	—	—	—	—	1	+11
+12	2	2	3	2	—	1	3	2	—	4	—	—	—	—	1	—	2	1	—	—	—	—	—	1	1	—	—	—	—	—	+12
+13	1	3	—	6	1	2	1	4	1	1	2	—	—	—	—	1	1	—	1	—	—	—	—	1	2	—	—	—	—	—	+13
+14	3	3	4	3	6	1	1	1	2	3	—	—	—	1	—	1	—	—	—	1	—	—	—	—	—	—	—	—	—	—	+14
+15	3	4	5	4	4	6	1	2	1	1	—	—	—	—	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	+15
+16	1	2	1	3	3	1	—	1	4	2	—	—	—	—	—	—	—	1	—	1	—	1	—	—	—	—	—	—	—	—	+16
+17	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	+17
+18	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	+18
+19	—	—	—	—	—	—	—	—	—	—	2	—	—	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	+19
+20	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	+20
+21	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	+21
+22	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	+22
+23	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	—	—	—	—	—	—	—	—	—	—	—	—	+23
+24	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	+24
+25	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	+25
**	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	**
+30	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	+30

III.

NOTE ON THE CORRELATION BETWEEN β_2 AND w'.

By R. C. GEARY, M.Sc.

In the present state of our knowledge, the problem of determining which of two tests of normality is the more efficient, can be dealt with only by numerical methods. Having obtained the probability points for the rival tests for normal samples of different sizes, we should calculate the values of each of the test functions for a large number of distributions actually found in practical work and ascertain which of the tests gives the greater number of values lying in the respective regions of improbability for normal samples. As between the β_2 and w' tests*, investigation must be confined to samples so large that one may have confidence in the probability points for β_2 for normal samples, determined by E. S. Pearson.

Dr Pearson's discovery, illustrated on Figs. 3 and 4 of his paper, of marked statistical relationships between β_2 and w' in samples drawn from different parent universes, has obviously an important bearing on this problem. If there were a rigid algebraical relation between the tests, they would be equally efficient. The analysis which follows is subject to the qualification that the statistical relationship may be closer than the coefficient of correlation indicates. The coefficient of correlation -1 will be found only when the relation is linear. Fig. 4 suggests that the relationship between β_2 and w' for samples of given size from a given parent population may approximate rather to a curvilinear type.

The only universes for which an algebraical relation subsists between β_2 and w' are those in which there are two categories only—measuring say 1 and 0. Suppose that the sample proportions are p (measuring 1) and q (measuring 0). Then

$$m_1 = p,$$

$$m_2 = pq,$$

$$m_4 = pq\,(1 - 3pq).$$

The mean deviation $$d = 2pq.$$

Then $$w' = d/\sqrt{m_2} = 2\sqrt{pq},$$

and $$\beta_2 = m_4/m_2^2 = 1/pq - 3,$$

so that $$\beta_2 = 4/w'^2 - 3.$$

Accordingly as w' increases β_2 diminishes.

In large samples the sets of values of the elements which minimise β_2 maximise w', and vice versa. If the y_i represent the deviations from the sample mean, then

$$\beta_2 = \frac{n'\Sigma y_i^4}{(\Sigma y_i^2)^2} \text{ and } w' = \frac{\Sigma |y_i|}{\sqrt{n'\Sigma y_i^2}}.$$

* [At the desire of Mr Geary I have left w' standing throughout this section, although in the first section of his paper he has invariably used it with the appropriate subscript, here dropped. Ed.]

If the sample is so large that the y's may be considered independent*, the *minimum* value of β_2, namely 1, is found when

$$|y_1| = |y_2| = \ldots = |y_{n'}|,$$

which gives the *maximum* value 1 of w'. The maximum value of β_2, namely n', and the minimum value $\frac{1}{\sqrt{n'}}$ of w' are found for the following series of values of the y's:

$$y_i \neq 0,$$
$$y_k = 0 \ (k \neq i).$$

E. S. Pearson's Table I clearly indicates a tendency in the seven universes (to which the normal with $\beta_2 = 3$ and $w' = \cdot 798$ might be added to give an eighth) for a high β_2 to accompany a low w' and vice versa.

The correlation between β_2 and w' for normal samples may be determined as follows. If the sample of $n' = n+1$ is

$$x_1, x_2, \ldots, x_{n'},$$

then
$$w' = \frac{\Sigma |x_i - \bar{x}|}{\sqrt{n(n+1)}\, s} = \sqrt{\frac{n+1}{n}}\,\frac{d}{s},$$

and
$$\beta_2 = \frac{(n+1)\Sigma |x_i - \bar{x}|^4}{n^2 s^4} = \left(\frac{n+1}{n}\right)^2 \frac{m_4}{s^4},$$

with
$$s^2 = \frac{1}{n}\Sigma |x_i - \bar{x}|^2,$$

Indicating by square brackets, [ ], universal means about fixed origin, the coefficient of correlation required is

$$r = \frac{[w'\beta_2] - [\beta_2][w']}{\sigma_{w'}\sigma_{\beta_2}}.$$

The values of $[\beta_2]$ and σ_{β_2} are known†:

$$[\beta_2] = \frac{3n}{n+2}; \quad \sigma^2_{\beta_2} = \frac{24(n+1)(n-1)(n-2)}{(n+2)^2(n+4)(n+6)}.$$

Since w' and s are independent,

$$[w'] = \sqrt{\frac{n+1}{n}}\,\frac{[d]}{[s]}, \quad [w'^2] = \frac{n+1}{n}\,\frac{[d^2]}{[s^2]},$$

with
$$[s] = e^{-\frac{1}{4n} + \frac{1}{24n^3} - \frac{1}{20n^5} + \frac{17}{112n^7} - \cdots}, \quad [s^2] = 1,$$

and
$$[d] = \sqrt{\frac{2n}{\pi(n+1)}}; \quad [d^2] = \frac{n}{(n+1)^2}\left(1 + \frac{2}{\pi}\sqrt{n^2-1} + \frac{2}{\pi}\sin^{-1}\frac{1}{n}\right).$$

The last of these formulae is due to Helmert‡. The value of $\sigma_{w'}$ will be found from

$$\sigma^2_{w'} = [w'^2] - [w']^2.$$

* A solution of the problem of determining the sets of values of the n' variates which give limiting values of β_2 and w' for all sizes of samples, and the respective limiting values, would have some mathematical interest.

† Readily derivable from formulae given by R. A. Fisher, *Proc. Royal Society, London*, Series A, 130 (1931), p. 25.

‡ *Astronomische Nachrichten*, Bd. 88, No. 2096 (1876).

There remains the calculation of

$$[w'\beta_2] = \left(\frac{n+1}{n}\right)^{\frac{5}{2}} \frac{[dm_4]}{[s^5]},$$

in which

$$[s^5] = \frac{(n+1)(n+3)}{n^2}[s].$$

Also

$$[dm_4] = \frac{1}{(n+1)^2}\{(n+1)[y_1^5] + n(n+1)[y_1^4 y_2]\},$$

with

$$y_1 = |x_1 - \bar{x}| \text{ and } y_2 = |x_2 - \bar{x}|.$$

The distribution of y_1 is

$$\frac{1}{\sigma}\sqrt{\frac{2}{\pi}}\, e^{-y_1^2/2\sigma^2}\, \delta y_1 \quad (\infty \geqslant y_1 \geqslant 0),$$

with

$$\sigma^2 = \frac{n}{n+1},$$

so that

$$[y_1^5] = 8\sigma^5\sqrt{\frac{2}{\pi}}.$$

The joint distribution of y_1 and y_2 is

$$\frac{1}{\pi\sigma^2\sqrt{1-\rho^2}}\left\{e^{-\frac{1}{2\sigma^2(1-\rho^2)}(y_1^2+y_2^2-2\rho y_1 y_2)} + e^{-\frac{1}{2\sigma^2(1-\rho^2)}(y_1^2+y_2^2+2\rho y_1 y_2)}\right\}\delta y_1 \delta y_2.$$

The value of ρ, the coefficient of correlation between $x_1 - \bar{x}$ and $x_2 - \bar{x}$, is $-\frac{1}{n}$. From this distribution it has been found that the value of $[y_1^4 y_2]$ is

$$\sigma^5\sqrt{\frac{2}{\pi}}(3 + 6\rho^2 - \rho^4).$$

All the data are now available. Upon reduction the coefficient of correlation r between w' and β_2 for normal samples takes the simple form

$$r = -\sqrt{\frac{2}{\pi}}\,\frac{(n-1)(n-2)}{[s]\,n(n+2)(n+3)\,\sigma_{w'}\sigma_{\beta_2}}.$$

The values of $[s]$, $\sigma_{w'}$ and σ_{β_2} in the denominator have been given above. The presence of the factors $n-1$ and $n-2$ in the numerator might have been anticipated. If either were missing, r would assume the impossible value ∞ in samples of 2 or 3, since the factors $\sqrt{n-1}$ and $\sqrt{n-2}$ occur in σ_{β_2} in the denominator. In samples of 2, of course, both β_2 and w' assume the same value 1 for all samples, while β_2 assumes the same value $\frac{3}{2}$ for all samples of 3.

When n tends towards infinity the coefficient of correlation tends towards

$$r_\infty = -\frac{1}{\sqrt{12(\pi-3)}} = -\cdot 7673\ldots.$$

The values of r for certain sizes of sample are as follows:

Size of Sample, n'... ...	6	11	26	51	101
Coefficient of Correlation, r	$-\cdot 921$	$-\cdot 851$	$-\cdot 809$	$-\cdot 790$	$-\cdot 778$

ON THE VALIDITY OF A CERTAIN PEARSON'S FORMULA.

By JAN WIŚNIEWSKI.

§ 1. In the memoir "On the Measurement of the Influence of broad Categories on Correlation" printed in Vol. IX. of *Biometrika* (pp. 116 to 139) Karl Pearson makes some statements which seem to need qualification.

The notations are as follows: variables x, y; the class-marks of the corresponding class intervals C_x, C_y. The first statement we shall consider is the assumption that in each class-interval the actual arithmetic average of the variable x or y is equal to the corresponding class-mark C_x or C_y. Of course, this is an approximation only and within its degree of accuracy only holds the ensuing formula (i) of the paper referred to:

$$r_{xC_x} = \frac{\sigma_{C_x}}{\sigma_x} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

It is more than sure that Karl Pearson was aware of the approximative character of form (i) when proposing it; we merely point to this circumstance in connection with the numerical example that will follow.

The main point of the present note is to show the limited validity of another formula of the memoir discussed, viz. form (ix):

$$r_{xy} = \frac{r_{C_yC_x}}{r_{xC_x}r_{yC_y}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ix)}.$$

(ix) is based upon (iv) and (viii),

$$r_{xy} = \frac{r_{yC_x}}{r_{xC_x}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iv)},$$

$$r_{yC_x} = \frac{r_{C_yC_x}}{r_{yC_y}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(viii)}.$$

The derivation of (iv) is the following (to quote Pearson's words): "...since a given x will have a constant class-mark, the correlation of y and C_x for a constant x is zero; that is to say that the partial correlation coefficient

$$_x\rho_{yC_x} = \frac{r_{yC_x} - r_{xy}r_{xC_x}}{\sqrt{1-r^2_{xy}}\sqrt{1-r^2_{xC_x}}} = 0" \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(a)}.$$

§ 2. Formula (a) and its derivation need a thoroughgoing discussion. First, we shall make some observations concerning partial correlation in general. According to the writer's view, partial correlation may be given two interpretations: the "layer" interpretation and the "projection" interpretation.

First, we shall present the more usual "layer" interpretation. Suppose we are dealing with the variables u, v, t. The points having coordinates u_i, v_i, t_i are

distributed in a three-dimensional space. Now if we want to find the partial correlation between u and v we divide the space into layers by means of planes parallel to the u, v plane. In each layer the variability of t is greatly reduced so that we consider it negligible. Within each layer we make a projection of the u_i, v_i, t_i points on a plane parallel to the u, v plane and calculate the correlation found for the distribution of the projected points (of course the projection is made perpendicularly to the said plane). At last, we compute a kind of average* of the correlation coefficients in each layer and this is the partial correlation coefficient, provided the regressions of u on t and of v on t are linear. If they are not strictly so, the partial correlation coefficient calculated according to the familiar formulae is at any rate an approximation to the coefficient calculated in the manner described above.

The "projection" interpretation is arrived at from the former when we assume that layers are made thinner and thinner. Instead of making a projection within each layer we make one of all u_i, v_i, t_i points on the u, v plane. But the projection is no more perpendicular to this plane. It is made in such a manner that the coordinates of each projected point are $u_i - \bar{u}_i$, $v_i - \bar{v}_i$ where $\bar{u}_i$ and $\bar{v}_i$ are values obtained from regression equations† on t. If we assume those equations to be linear, we again get the classical formula $(r_{uv} - r_{ut}r_{vt})/\sqrt{(1-r^2_{ut})(1-r^2_{vt})}$ equivalent to the simple correlation coefficient calculated for the projected points. If t is a discrete variable and the layers are made so thin as to comprise only one value of t, the "layer" and the "projection" interpretations are identical.

§3. Now we can analyse the special case of ${}_x\rho_{yC_x}$. The distribution of the y_i, C_{xi}, x_i points is very peculiar indeed. Assume that x and y are continuous variables, i.e. they can take any given value within the field of variation. C_x, on the other hand, can take only special values. Moreover, C_x is a definite function of x, viz.

$$C_x = h\left[\frac{x}{h}\right] + \frac{h}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (b),$$

where $[z]$ means the largest integer smaller than z, and h is the class-interval.

For this reason all y_i, C_{xi}, x_i points must be situated on plane "stripes" parallel to the x, y plane. Each "stripe" is bounded by two parallel straight lines which are determined as follows:

$$C_x = C_{xi};\ x = C_{xi} + \tfrac{1}{2}h \quad \text{and} \quad C_x = C_{xi};\ x = C_{xi} - \tfrac{1}{2}h.$$

The "layer" interpretation fails in this case completely. As a matter of fact it is only natural to take x-layers identical with its class-intervals. Then within each

* This kind of average resembles what Irving Fisher terms "aggregatic average." In our case, denoting with $\bar{u}_k$ and $\bar{v}_k$ the arithmetic averages of u and v for the layer k, we have

$$\rho = \frac{\sum\limits_k \sum\limits_i (u_i - \bar{u}_k)(v_i - \bar{v}_k)}{\sqrt{\sum\limits_k \sum\limits_i (u_i - \bar{u}_k)^2 \sum\limits_k \sum\limits_i (v - \bar{v}_k)^2}}.$$

† In general, these equations need not be *perfectly* fulfilled.

layer all points will lie on a straight line parallel to the y-axis. This means that the correlation coefficient for each layer becomes indefinite and so is the partial correlation coefficient calculated according to the "layer" interpretation.

We would obtain the same result from the "projection" interpretation if we assumed that the values of $\overline{C}_{xi}$ are determined from equation (b), which is really the case. By $\overline{C}_{xi}$ is meant the value of C_{xi} obtained from a regression equation.

It seems that the only way to get for ${}_x\rho_{yC_x}$ a definite value is to start from the "projection" interpretation and to assume

$$\overline{C}_{xi} = r_{xC_x}\frac{\sigma_{C_x}}{\sigma_x}(x_i - M_x) + M_{C_x} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(c),$$

where M_x and M_{C_x} denote the arithmetic averages of x and C_x respectively. As we stated, each y_i, C_{xi}, x_i point must lie on a "stripe" conforming to the description given above. More particularly, for a given x_i all points lie on a straight line which is defined by $x = x_i$, $C = C_{xi}$ as determined from (b). Such a straight line is, of course, situated on a given "stripe" and parallel to its boundaries. What will be the projection of the points lying on such a line, on the y, C_x plane? The projections will have a common C_x coordinate equal to $C_{xi} - \overline{C}_{xi}$, where $\overline{C}_{xi}$ is determined from (c), and y's equal to $y_i - \overline{y}_i$, where $\overline{y}_i$ is found from the regression equation of y on x. As $C_{xi} - \overline{C}_{xi}$ takes different values* the projected points will not lie on one straight line parallel to the y-axis (as in the "layer" interpretation case). If y is not a one-valued function of x, the projected points also will not lie on one straight line parallel to the C_x-axis and, consequently, ${}_x\rho_{yC_x}$ will take a definite value. If the regression of y and x is strictly linear, the mathematical expectation of $y_i - \overline{y}_i$ for each $C_{xi} - \overline{C}_{xi}$ is zero and therefore ${}_x\rho_{yC_x}$ will equal zero. If the regression is only "practically" linear, as is assumed by Pearson, equation (a) still will give a good approximation.

The conclusion is that (iv) is true provided the regression of y on x is linear. This is a sufficient condition but not absolutely necessary. We can fancy cases where the regression in question is not strictly linear and yet ${}_x\rho_{yC_x} = 0$.

§4. The derivation of (viii) is similar; namely, Pearson assumes that the correlation of C_x and C_y for a constant y is zero and, accordingly,

$${}_y\rho_{C_xC_y} = \frac{r_{C_xC_y} - r_{yC_x}r_{yC_y}}{\sqrt{1 - r^2_{yC_x}}\sqrt{1 - r^2_{yC_y}}} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(d).$$

In order to examine the validity of (d) we shall start from the "projection" interpretation and assume that $\overline{C}_{xi}$ and $\overline{C}_{yi}$ are determined from equations analogous to (c).

The distribution of the C_{xi}, C_{yi}, y_i points in space is still more peculiar than in the foregoing case. Each point lies on a rectilinear segment parallel to the y-axis and limited by two points, having the coordinates $C_x = C_{xi}$; $C_y = C_{yi}$;

* If x is a continuous variable $C_{xi} - \overline{C}_{xi}$ is also one. For practical purposes we may assume, however, that $|C_{xi} - \overline{C}_{xi}| < h$.

$y = C_{yi} \pm \frac{1}{2}h$ (supposing the class-intervals of x and of y to be the same). For a given C_{yi} all such segments lie on a plane "stripe" analogous to those described in the preceding paragraph. The difference is that in the former case the points lay distributed more or less evenly throughout the stripe while in the present one they can occupy only determined positions on rectilinear segments parallel to the y-axis.

Correspondingly the projection of the C_{xi}, C_{yi}, y_i points on the C_x, C_y plane will differ from that of the former case. Let us denote, for the sake of simplicity,

$$u_i = C_{yi} - \bar{C}_{yi} = C_{yi} - M_{C_y} - r_{yC_y}\frac{\sigma_{C_y}}{\sigma_y}(y_i - M_y) \quad \ldots\ldots\ldots\ldots\ldots(e),$$

$$v_i = C_{xi} - \bar{C}_{xi} = C_{xi} - M_{C_x} - r_{yC_x}\frac{\sigma_{C_x}}{\sigma_y}(y_i - M_y) \quad \ldots\ldots\ldots\ldots\ldots(f),$$

from which follows

$$v_i = C_{xi} - M_{C_x} + \frac{r_{yC_x}\sigma_{C_x}}{r_{yC_y}\sigma_{C_y}}\{u_i - (C_{yi} - M_{C_y})\} \quad \ldots\ldots\ldots\ldots\ldots(g).$$

(g) is evidently linear with respect to u. For the whole stripe we have a constant C_y and for each segment within a stripe a constant C_x. Therefore the projection of the segments of a stripe will be a set of segments parallel to each other and having the same field of variation of u (because the field of variation of y is the same for all segments, viz. from $C_y - \frac{1}{2}h$ to $C_y + \frac{1}{2}h$). As r_{yC_y} is positive and near to unity, the angular coefficients of the projected segments will depend mainly on r_{yC_x}.

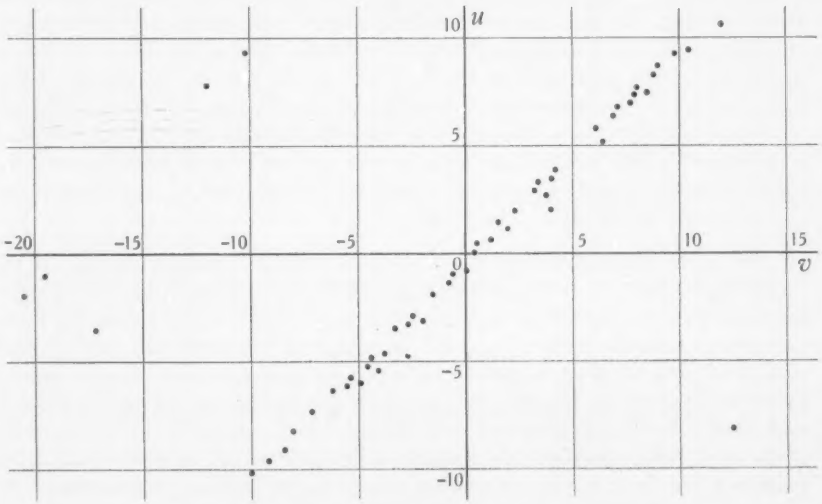


Chart 1. Distribution of observations from a numerical example.

Data projected according to equations (e) and (f). The dots on this chart do not number 75, as they should, because some of them represent several observations.

All segments of our distribution will also give a set of parallel segments*, but the segments taken from different stripes will have fields of variation of u of the same length, to be sure, yet somewhat shifted—suppose both C_{yi} and y_i in (e) increase by h, then u_i will increase by $h\left(1 - r_{yC_y}\frac{\sigma_{C_y}}{\sigma_y}\right)$.

What will be the correlation calculated for the projected distribution? Off-hand we cannot say anything. If the points were distributed evenly in each segment, the correlation would surely follow the sign of r_{yC_x}, the absolute value of ρ approaching zero as the number of segments for each C_y would increase. But the premise given above is not general; indeed, if we really have to do with "broad categories," we might expect that the correlation between y and C_x will be observable already within given y-intervals, i.e. for a given C_y. The coefficient of correlation for the points thus projected is evidently

$$_y\rho_{C_yC_x} = \frac{r_{C_yC_x} - r_{yC_y}r_{yC_x}}{\sqrt{1 - r^2_{yC_y}}\sqrt{1 - r^2_{yC_x}}} \quad \text{.....}(h).$$

This cannot be zero except when

$$r_{yC_x} = r_{C_yC_x} = 0 \quad \text{.....}(j),$$

or

$$\frac{r_{C_yC_x}}{r_{yC_x}} - r_{yC_y} = 0 \quad \text{.....}(k).$$

Thus we are brought back to our starting point. (k) is clearly equivalent to (viii). Therefore we see that Pearson's contention (viii) cannot be generally proved in an analytical way. In the best case we can regard (viii) as an approximation of unknown precision. The application of the reasoning used at the end of § 3 is here impossible as the regression of C_x on y will *not*, in general, be strictly linear; consequently, the mathematical expectation of $C_{xi} - \overline{C}_{xi}$ for a given $C_{yi} - \overline{C}_{yi}$ will not necessarily equal zero. Especially, if some observations are distributed in such a manner that in a certain number of y-rows one cell only is filled in each, the aforesaid mathematical expectation cannot be generally zero. In such cases the sign of (h) will surely follow that of $r_{C_yC_x}$.

The above conclusions can also be derived from a discussion of (k). If the distribution is such as just described $r_{C_yC_x}$ should nearly equal r_{yC_x}. On the other hand, we may expect that in such cases the classification is really done in "broad categories," thus the value of r_{yC_x} will be comparatively small and the left-hand side of (k) will be positive instead of zero. We may also resort to the graphic presentation given in Chart 2. Suppose in some y-rows one cell only is filled in each, then this will be surely that cell through which passes the line of regression of C_x on y, this regression being assumed linear. The points of this cell, when projected, will lie on a segment passing near to the (0, 0) point. If the bulk of the

* Such a projection is shown in Chart 1. It can be clearly seen that the projected points form a set of parallel segments. [Surely any reasonable test for outliers would throw out the five points to the left and the single point to the right of Chart 1? Ed.]

observations belongs to such cells we shall have a picture like that in Chart 2—viz. a great mass of points situated on a narrow stripe passing near to the origin of the system. Those points will clearly contribute to produce a correlation coefficient of large numerical value and of the sign of $r_{C_y C_x}$. If, on the other hand, the majority of points belong to other cells, their projections will give a picture of the scattered points in Chart 2, those points contributing to lower the numerical value of ${}_y\rho_{C_y C_x}$.

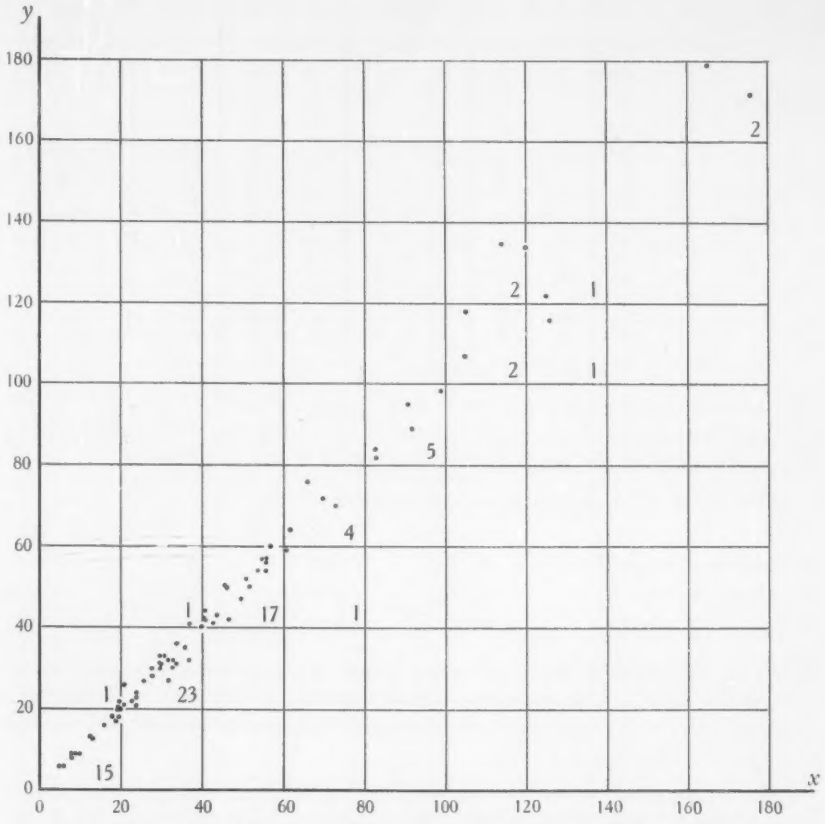


Chart 2. Distribution of observations from the numerical example. Original data.

It should also be noticed that our conclusions would be unchanged if instead of (e) we simply took $u_i = C_{yi} - y_i$.

§ 5. To sum up: Pearson's formulae (viii) and (ix) are not generally true, and their degree of approximation is unknown. How serious blunders can arise, in extreme cases, from their application, the following example will show:

Numerical Example. We will study the distribution of rural districts in a certain Polish county according to the number of dwelling houses (*Statistique de la Pologne* publiée par l'Office Central de Statistique de la Rép. Polonaise, Série B, fasc. 8–b, pp. 11—12. Arrondissement de Chodzież, communes rurales. Nombre de bâtiments d'habitation). x-data from 1921 census, y-data from 1931 census. The original data are given in the table below:

x	y	x	y	x	y	x	y	x	y
23	22	5	6	33	30	19	17	36	35
41	42	56	57	62	64	40	40	30	31
52	50	41	44	41	42	28	28	21	26
30	30	66	76	8	9	83	82	31	33
24	24	13	13	37	32	73	70	114	135
83	84	18	18	120	134	54	54	30	31
34	36	20	20	34	31	55	57	105	118
20	22	9	9	26	27	46	50	13	13
37	41	20	20	32	27	30	33	70	72
20	18	28	30	176	172	46	50	33	32
10	9	8	8	44	43	105	107	56	56
91	95	56	54	57	60	99	98	24	23
51	52	61	59	6	6	50	47	125	122
32	32	165	179	43	41	24	21	92	89
126	116	16	16	47	42	20	20	21	21

The data are also charted and given in graphic form (Chart 2). The cross-classification has been executed in intervals of twenty houses, including the upper class-limit. The figures given in the graph are the frequencies in the several cells.

The values of the parameters of the distribution are:

	x	C_x	y	C_y
Arithmetic average M	47·93	48·10	48·71	48·63
Standard deviation σ	35·92	36·66	37·45	36·84

$N = 75$, $r_{xy} = ·9934$, $r_{C_xC_y} = ·9879$, $r_{xC_x} = ·9871$, $r_{yC_y} = ·9877$, $r_{yC_x} = ·9810$.

All r's are computed directly, i.e. making no use of the quoted Pearson's formulae. Now in the light of this example we shall test the validity both of the formulae mentioned and of the limitations we suggested. The particular example we chose seems to be well adapted for this purpose.

M_x and M_{C_x} differ widely, so formula (i) cannot be expected to give good results. Indeed, r'_{xC_x}* actually comes out greater than 1! On the other hand, the difference between M_y and M_{C_y} is rather slight and correspondingly $r'_{yC_y} = ·9837$, which is not far from the value calculated directly.

As the regression of y on x is very nearly linear, form (iv) should give correct results. As a matter of fact, $r_{yC_x}/r_{xC_x} = ·9938$. This discrepancy between this value

* We denote by r' those coefficients which are calculated indirectly, i.e. by means of Pearson's formulae.

and the one given above is within the margin of error due to rounding off decimals.

Because of the existence of many y-rows containing only one cell that is not empty, formulae (viii) and (ix) must register failures. Thus ${}_x\rho_{\sigma_y\sigma_x}$ instead of zero equals ·626, $r_{\sigma_y\sigma_x}/(r_{x\sigma_x}r_{y\sigma_y}) > 1$; $r_{\sigma_y\sigma_x}/(r'_{x\sigma_x}r'_{y\sigma_y}) = ·9919$ which happens to be in the right direction, but we wonder whether one would not hesitate to use r''s greater than unity.

Lastly, we shall test Pearson's formula (xv) devised for the case when, as usually, we know very little about the distribution within class intervals.

$$r''^2_{x\sigma_x} = 1 - \frac{h_x^2}{12\sigma_x^2} \quad \text{.....(xv),}$$

where h is the class-interval. Here again $r_{\sigma_y\sigma_x}/(r''_{x\sigma_x}r''_{y\sigma_y}) > 1$.

One might object that the differences between theory and application are so minute as not to deserve attention, but the corrections to crude coefficients as proposed by Pearson are themselves, in general, of a rather small order.

ON THE CORRECTIONS FOR BROAD CATEGORIES, BEING A NOTE ON MR WIŚNIEWSKI'S MEMOIR.

By K. PEARSON.

I PROPOSE to discuss Mr Wiśniewski's criticism of my method of correcting for broad categories, by dealing first with the numerical example which he provides. I may state that I consider that example a very extreme one to test the accuracy of my corrections upon, and the reasons for this will appear in the course of my analysis.

Mr Wiśniewski's data consist of the numbers of inhabited houses in certain rural districts or arrondissements in Poland at the date of two censuses. There is no district without a house, the lowest number being five, and the distributions at both censuses are crowded into the ranges 5 to 60. Thus the frequency distributions of the two variates are both extremely skew, we are ignorant of where they start, and the sample consists of 75 rural districts only, so that the standard errors of the constants involved are very large, and no allowance is made for any of these peculiarities. The case is, however, said to be singularly suitable for testing my formulae.

In Table I the actual data are given, where I must remark that the groups of 20 houses are included in the group 0 to 20, and the group 20 to 40 does not include the 20 and does include the 40 and so on.

Hence, if this discrete series is to be replaced by a continuous variate we must take our groups to be 20·5 to 40·5 and so on. Mr Wiśniewski does not appear to notice that my formulae are deduced on the assumption of *continuous* and not discrete variables. Even when we replace discrete by continuous variables the start of the continuous variables is left in the air. Now I show in my original paper that my correction for the correlation coefficient reduces to Sheppard's correction, when the sub-ranges are all equal, as they are in this case (except the doubtful first category) and when we accept the hypothesis of Sheppard, i.e. *that high contact at the terminals of the distribution holds.* Now in Mr Wiśniewski's data there is no high contact at the zero ends of the frequency distributions, but on the contrary great abruptness. Hence the standard deviations involved in the value of the correlation must be corrected in a manner different from that involved in taking

$$r_{x\sigma_x} = \sqrt{1 - \frac{h^2}{12\sigma_x^2}},$$

a relation which depends on the high contact hypothesis. It is overlooking this point which leads him to obtain a corrected correlation coefficient slightly over unity.

TABLE I

Values 1921 *Census*

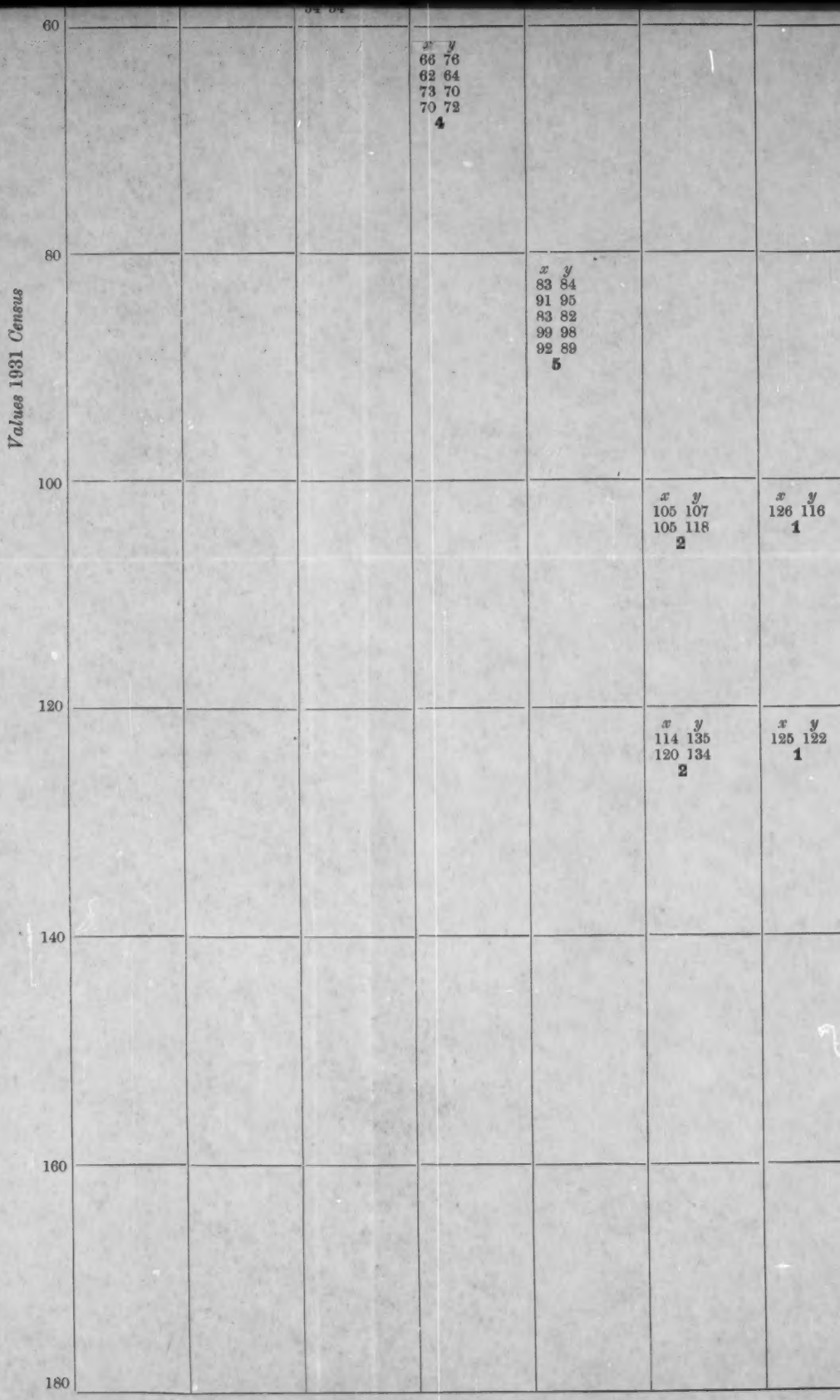
Values 1931 *Census*

	0–20	20–40	40–60	60–80	80–100	100–120	120–140
0–20	*x* *y* 10 9 5 6 13 13 18 18 20 18 20 20 9 9 20 20 8 8 16 16 8 9 6 6 *x* *y* 19 17 20 20 13 13 **15**						
20–40	*x* *y* 20 22 **1**	*x* *y* 23 22 30 30 24 24 34 36 21 26 32 32 28 30 33 30 37 32 34 31 21 21 26 27 *x* *y* 32 27 40 40 28 28 30 33 24 21 36 35 30 31 31 33 30 31 33 32 24 23 **23**					
40–60		*x* *y* 37 41 **1**	*x* *y* 41 42 52 50 51 52 56 57 41 44 56 54 41 42 44 43 57 60 43 41 47 42 54 54 *x* *y* 55 57 46 50 46 50 50 47 56 56 **17**	*x* *y* 61 59 **1**			
60–80				*x* *y* 66 76 62 64 73 70 70 72 **4**			
80–100					*x* *y* 83 84 91 95 83 82 99 98 92 89 **5**		
100–120						*x* *y* 105 107 105 118 **2**	*x* *y* 126 116 **1**

140 160 180

Ordinary figures give the coordinates of the individuals in each cell.
Heavy type gives the total number of individuals in each cell.

x y 126 116 **1**		


Values 1931 Census
60
80
100
120
140
160
180
x y
66 76
62 64
73 70
70 72
4
x y
83 84
91 95
83 82
99 98
92 89
5
x y
105 107
105 118
2
x y
126 116
1
x y
114 135
120 134
2
x y
125 122
1

To face p. 365

	x 126 y 116 1		
	x 125 y 122 1		
			x 165 y 179 176 172 2

Taking the data as given in Table I and working with the 75 individual cases, I find, x being the horizontal and y the vertical variate,

Means $\bar{x} = 47{\cdot}933{,}333, \quad \bar{y} = 48{\cdot}706{,}667,$

Standard Deviations

$$\sigma_x = 35{\cdot}914{,}466, \quad \sigma_y = 37{\cdot}454{,}959,$$

Product Moment Coefficient $= 1336{\cdot}353{,}777,$

providing the correlation coefficient

$$r_{xy} = {\cdot}993{,}442.$$

Mr Wiśniewski gives the values

$$\bar{x} = 47{\cdot}93, \quad \bar{y} = 48{\cdot}71, \quad \sigma_x = 35{\cdot}92, \quad \sigma_y = 37{\cdot}45, \quad r_{xy} = {\cdot}9934,$$

which nearly agree with the above values, although they are taken to an inadequate number of figures having regard to later work.

Now the problem before us is to obtain these results, or results close to them from a contracted table, i.e. one of broad categories, and so save the tedious process of dealing with the individual 75 entries in computing the coefficients. But the corrections for grouping observations together have hitherto been based on *continuity* of the variates, except in those proposed by H. C. Carver for grouping discrete variates*, and these fail in the present case, as we shall see, precisely as Sheppard's fail, and for the same reason—they disregard the abruptness. In order accordingly to correct the contracted table we are bound to suppose the frequencies represented on it result from the sampling of a *continuous* distribution. In the next place we are at liberty to choose our grouping, and the safest grouping experience shows is that which starts close to the lowest observation in the data; in this case 5, 5. Let us take our grouping accordingly in sub-ranges of 20 from 4·5. We then reach Table II.

TABLE II.

x-variate

y-variate	*4·5—24·5*	*24·5—44·5*	*44·5—64·5*	*64·5—84·5*	*84·5—104·5*	*104·5—124·5*	*124·5—144·5*	*144·5—164·5*	*164·5—184·5*	Totals
4·5—24·5	21	—	—	—	—	—	—	—	—	21
24·5—44·5	1	23	1	—	—	—	—	—	—	25
44·5—64·5	—	—	13	—	—	—	—	—	—	13
64·5—84·5	—	—	—	5	—	—	—	—	—	5
84·5—104·5	—	—	—	—	3	—	—	—	—	3
104·5—124·5	—	—	—	—	—	2	2	—	—	4
124·5—144·5	—	—	—	—	—	2	—	—	—	2
144·5—164·5	—	—	—	—	—	—	—	—	—	—
164·5—184·5	—	—	—	—	—	—	—	—	2	2
Totals	22	23	14	5	3	4	2	—	2	75

* *Annals of Mathematical Statistics*, Vol. I. p. 111.

The peculiar nature of the distribution upon which Mr Wiśniewski proposes to test my corrections will now be obvious to the reader. There is extreme abruptness at the low end of the range for both variates, and the usual, or Sheppard's corrections, are bound to fail. However, let us see how the results work out. We will take our working axes through 54·5, 54·5, and our working unit 20 houses. Summations being from working origin. We find

$\frac{1}{75}S(x) = -\cdot 32$, $\frac{1}{75}S(y) = -\cdot 32$, $\frac{1}{75}S(x^2) = 3\cdot 573{,}333$, $\frac{1}{75}S(y^2) = 3\cdot 546{,}666$,

and $\frac{1}{75}S(xy) = 3\cdot 52$, and $q_{xy} = 3\cdot 52 - \cdot 1024 = 3\cdot 4176$.

Hence, making no corrections, we have

$$\sigma_x^2 = 3\cdot 470{,}933, \qquad \sigma_y^2 = 3\cdot 444{,}267, \qquad q_{xy} = \frac{S(xy)}{75} - \bar{x}\bar{y} = 3\cdot 4176,$$

$$\sigma_x = 1\cdot 863{,}044, \qquad \sigma_y = 1\cdot 855{,}874, \qquad \sigma_x\sigma_y = 3\cdot 457{,}575.$$

Accordingly the crude uncorrected r_{xy} is given by

$$r_{xy} = q_{xy}/\sigma_x\sigma_y = \cdot 988{,}438.$$

Now Sheppard has shown that if the sub-ranges are equal and sufficiently small as compared with σ_x and σ_y* and there is high contact at both ends for continuous variates, then q_{xy} requires no correction and $\frac{1}{12}$ (sub-range)^2 must be subtracted from σ_x^2 and σ_y^2. I have shown that under like conditions my class index corrections $r_{x\mathcal{C}_x}$ and $r_{y\mathcal{C}_y}$ reduce to $\sqrt{1 - \frac{h^2}{12\sigma_x^2}}$ and $\sqrt{1 - \frac{k^2}{12\sigma_y^2}}$, where h and k are the x and y sub-ranges respectively. These lead if the conditions stated above be supposed to hold to precisely the same result as Sheppard's corrections.

Using Sheppard, we find

$$\sigma_x = 1\cdot 840{,}5433, \qquad \sigma_y = 1\cdot 833{,}2850, \qquad \sigma_x\sigma_y = 3\cdot 374{,}240,$$

and accordingly

$$r_{xy} = 3\cdot 4176/3\cdot 374{,}240 = 1\cdot 012{,}85.$$

Thus these corrections fail, and the reason for failure lies not so much in the fact that h is more than half σ_x, as in the absence of high contact at the start of the ranges of both x and y. Before we turn to the correction for abruptness, let us see what Carver's formulae will provide.

Now if there be u discrete values in the sub-range, Carver subtracts

$$\frac{1}{12}\left(1 - \frac{1}{u^2}\right) = \frac{1}{12}\left(1 - \frac{1}{400}\right)$$

from the standard deviation squared, i.e. ·0831,2500 instead of ·0833,3333, like Sheppard. This gives us

$$\sigma_x = 1\cdot 840{,}5999, \qquad \sigma_y = 1\cdot 833{,}3418, \qquad \text{and } \sigma_x\sigma_y = 3\cdot 374{,}4487.$$

Thus the corrected $r_{xy} = 3\cdot 4176/3\cdot 374{,}4487 = 1\cdot 012{,}78.$

* This is certainly not fulfilled in the present data.

We see again that the correlation is greater than unity, and with twenty discrete values in the sub-range Carver's formula scarcely improves on Sheppard's, which treats them as continuous. Thus the failure does not arise from treating discrete grouping as continuous.

The source of error clearly lies in neglecting the abruptness. Allowance for abruptness has been made by Pairman and Pearson* and put into a convenient form by Sandon†, but possibly the best summary is that provided in Part II, p. cxciii of the *Tables for Statisticians and Biometricians*. Assuming high contact at the end of the range and moments to be taken about the start, we have to correct the means and the standard deviations of the two variates by the formulae

$$\mu_1' = \nu_1' - hH_{1,a}, \qquad \mu_2' = \nu_2' - \tfrac{1}{12}h^2 - h^2H_{2,a},$$

where
$$H_{1,a} = \frac{11{,}153n_1 - 12{,}566n_2 + 10{,}176n_3 - 4{,}586n_4 + 863n_5}{60{,}480N},$$

$$H_{2,a} = \frac{1{,}830n_1 - 4{,}358n_2 + 4{,}110n_3 - 1{,}962n_4 + 380n_5}{60{,}480N}.$$

Here n_1, n_2, n_3, n_4 and n_5 are the first five sub-frequencies from the abrupt terminal, and N is the total frequency $= 75$ in our case. Noting, that for x,

$$n_1 = 22, \quad n_2 = 23, \quad n_3 = 14, \quad n_4 = 5, \quad n_5 = 3,$$

and that of y, $n_1 = 21$, $n_2 = 25$, $n_3 = 13$, $n_4 = 5$, $n_5 = 3$,

we find
$${}_xH_{1,a} = +\cdot0172{,}9960, \qquad {}_xH_{2,a} = -\cdot0024{,}4797,$$
$${}_yH_{1,a} = +\cdot0070{,}5688, \qquad {}_yH_{2,a} = -\cdot0056{,}7901.$$

Accordingly,
$${}_x\mu_1' = 2{\cdot}5 - {\cdot}32 - \cdot0172{,}9960 = 2{\cdot}1627{,}0040,$$
$${}_y\mu_1' = 2{\cdot}5 - {\cdot}32 - \cdot0070{,}5688 = 2{\cdot}1729{,}4312.$$

The corresponding values from the origin (2·5, 2·5) are

$${}_x\mu_1'' = -\cdot3372{,}9960, \qquad {}_y\mu_1'' = -\cdot3270{,}5688,$$

and accordingly the corrected q_{xy} is given by

$$q_{xy} = \tfrac{1}{75}S(xy) - {}_x\mu_1'' \times {}_y\mu_1'' = 3{\cdot}52 - \cdot1103{,}1615 = 3{\cdot}4096{,}8385.$$

Again
$${}_x\nu_2' = \tfrac{1}{75}S(x^2) + 5\frac{S(x)}{75} + 6{\cdot}25 = 3{\cdot}5733{,}3333 - 1{\cdot}60 + 6{\cdot}25 = 8{\cdot}2233{,}3333,$$
$${}_x\mu_2' = 8{\cdot}2233{,}3333 - \cdot0833{,}3333 - (-\cdot0024{,}4797) = 8{\cdot}1424{,}4797,$$
$$\sigma_x^2 = {}_x\mu_2' - {}_x\mu'^2 = 8{\cdot}1424{,}4797 - 4{\cdot}6772{,}7302 = 3{\cdot}4651{,}7495,$$
and
$$\sigma_x = 1{\cdot}8614{,}9804.$$

Again
$${}_y\nu_2' = \tfrac{1}{75}S(y^2) + 5\frac{S(y)}{75} + 6{\cdot}25 = 3{\cdot}5466{,}6667 - 1{\cdot}60 + 6{\cdot}25 = 8{\cdot}1966{,}6667,$$
$${}_y\mu_2' = 8{\cdot}1966{,}6667 - \cdot0833{,}3333 + \cdot0056{,}7901 = 8{\cdot}1190{,}1234,$$
$$\sigma_y^2 = {}_y\mu_2' - {}_y\mu_1'^2 = 8{\cdot}1190{,}1234 - 4{\cdot}7216{,}8180 = 3{\cdot}3973{,}3054,$$
and
$$\sigma_y = 1{\cdot}8431{,}8489.$$

* *Biometrika*, Vol. XII. pp. 233, 239.
† *Biometrika*, Vol. XVI. pp. 193—195.

Thus $$\sigma_x \sigma_y = 3{\cdot}4310{,}8506,$$

and $$r_{xy} = \frac{q_{xy}}{\sigma_x \sigma_y} = \frac{3{\cdot}4096{,}8385}{3{\cdot}4310{,}8506} = {\cdot}993{,}163,$$

and this differs from the true value ·993,442 by ·000,321. Even this might have been got rid of had a more elaborate attempt been made to determine the probable start of the marginal frequency distributions.

Now let us examine what Mr Wiśniewski has done. He begins by assuming that my class-marks C_x and C_y are the midpoints of the sub-ranges. On the very first page of my memoir, I start by saying that I am going to take the mean $\bar{x}_s$ of the group of individuals in the sth class as my class-mark. I then proceed later in the memoir to discuss the conditions under which it is possible to replace the summations involving $\bar{x}_s$, $\bar{y}_t$ by corresponding summations of the midpoints x_s, y_t of the sub-ranges as class-marks. Mr Wiśniewski pays no attention to these conditions, never enquiring whether they are satisfied or not, and thus reaches corrected values of the correlation which exceed unity, which they may easily do as the uncorrected correlation of his suitable case is already ·988 and over. I can only think that he has not read, or if he has done, not understood the meaning of my memoir. On pp. 120 and 128 it is stated that the vanishing of certain summations depends upon high contact at the terminals or on the truth of Sheppard's hypothesis. When this hypothesis does not hold, then, as is well known, corrections have to be made for "abruptness."

It is idle to say how unusable these formulae are if we take as an illustration a case expressly excluded by the author of them. But in these excluded cases, if proper corrections are made for the means and standard deviations, we find that the short table gives a result close to the true value obtained from the full data.

Mr Wiśniewski never tells us what he means by a "class-mark." I understand by it a term which is common to every individual in a broad category, and any quantity may be used as the class-mark which is constant for every member in the group. Thus the mean x of the group may be used, or the mid-point of the sub-range, or the median of the values of x in the sub-range. I start my discussion by saying I will take the first of these as convenient, but pass on in the course of the paper to consider the conditions under which we may use other values to represent the "class-mark."

I now come to the criticism of my formulae and the degree of their approximation, using of course the value of C_x for which they are established.

In the first place, let us consider Formula (i) of the previous memoir

$$r_{xC_x} = \frac{\sigma_{C_x}}{\sigma_x} \quad \text{.....(i).}$$

Let x and C_x $(= \bar{x}_s)$ be measured from the mean of the variate x, and let n_s be the frequency in the sth class, Σ a summation within the class, and S a summation for all classes. Then by definition

$$r_{xC_x} = \frac{S(C_x \Sigma x_s)}{N \sigma_x \sigma_{C_x}} = \frac{S(C_x n_s C_x)}{N \sigma_x \sigma_{C_x}} = \frac{S\left(\frac{n_s}{N} C_x^2\right)}{\sigma_x \sigma_{C_x}} = \frac{\sigma_{C_x}^2}{\sigma_x \sigma_{C_x}} = \frac{\sigma_{C_x}}{\sigma_x}.$$

The formula is thus *absolutely* true and not approximative at all, and Mr Wiśniewski's remark "It is sure that Karl Pearson was aware of the approximative character of this formula (i)" appears somewhat wide of the mark.

Let us turn now to a second formula for the partial correlation of

$$_{x}r_{y\sigma_x} = 0.$$

Mr Wiśniewski gives a somewhat lengthy discussion of the meaning of a partial correlation coefficient*. I should interpret the relationship here with discrete x and y to be as follows: If we pick out for a given x all the corresponding y's and C_x's, then the product moment $S(y-\bar{y}_x)(C_x-(\bar{C}_x)_{x=\text{const.}})$ must vanish. This it clearly does. For C_x is a constant when x is constant, and $S(y)=n_x\bar{y}_x$, both factors of the product being independent for constant x. There is accordingly, in my mind, no doubt of the absolute truth of

$$_{x}r_{y\sigma_x} = 0.$$

In the same manner

$$_{xy}r_{\sigma_x\sigma_y} = 0$$

is absolutely true, for the correlation of two constant quantities is zero. The question then arises as to how far we may replace these partial correlation coefficients by the customary expansion formulae for linear regression. This is discussed at considerable length in my memoir, and involves the relationship between the three following summations

$$S\left(\frac{n_{st}}{N}\bar{x}_{st}\bar{y}_{st}\right), \qquad S\left(\frac{n_{st}}{N}x_s y_t\right), \text{ and } \quad S\left(\frac{n_{st}}{N}\bar{x}_s\bar{y}_t\right),$$

where n_{st} is the frequency in the n_{st} cell, i.e. that common to the sth column and tth row, x_s, y_t are the midpoints of the sth column and tth row, and $\bar{x}_s$, $\bar{y}_t$ are the means of the x-constants of the sth column and of the y-constants of the tth row, namely, they are what I term on the first page of my memoir C_x and C_y, the class indices.

On p. 123, the fundamental equation (xvii) is reached, namely†,

$$r_{xy} = \frac{S\left(\frac{n_{st}}{N}\frac{\bar{x}_s}{\sigma_x}\frac{\bar{y}_t}{\sigma_y}\right)}{S\left(\frac{n_t\bar{x}_s^2}{N\sigma_x^2}\right)\times S\left(\frac{n_t\bar{y}_s^2}{N\sigma_y^2}\right)} \quad \text{.....(xvii),}$$

or, in our present notation,

$$r_{xy} = \frac{S\left(\frac{n_{st}}{N}C_xC_y\right)}{\sigma_{C_x}\sigma_{C_y}\times\frac{\sigma_{C_x}}{\sigma_x}\frac{\sigma_{C_y}}{\sigma_y}} = \frac{r_{C_xC_y}}{r_{xC_x}r_{yC_y}} \quad \text{.....(xvii \textit{bis}),}$$

which amounts to saying

$$_{xy}r_{\sigma_x\sigma_y} = 0$$

when that partial correlation coefficient is expressed in the ordinary linear form‡.

* As far as I can follow Mr Wiśniewski, he is only indicating the difference long recognised by biometricians between 'singular' and 'plural' partial correlations.

† All quantities are for the remainder of this note to be measured from the means of the x and y variates.

‡ See footnote p. 118 of my memoir.

But what are the expressly stated conditions under which (xvii *bis*) will be a close approximation to the true result?

(i) The most important is high contact at both end terminals of the total ranges, i.e. cases in which the Sheppard corrections are adequate. This condition has been entirely forgotten or disregarded by Mr Wiśniewski, and he chooses as a suitable example a case where there is abruptness of a high order at one terminal of both the x and y distributions!

(ii) In deducing (xvii *bis*) we have nowhere assumed that the regressions of C_x or C_y on x or y are linear. The regression of x on C_x and of y on C_y are a series of points on 45° lines. The regressions of C_x and C_y on x and y respectively are staircases, the midpoints of the treads being on 45° lines, approaching more and more nearly to straight lines as the sub-ranges h and k are reduced. But these do not for the present concern us*. The only assumption of linearity made is that of x on y (see top of p. 123 of my memoir). By the nature of things this must be closely fulfilled in Mr Wiśniewski's example.

(iii) Is there any other assumption made in deducing (xvii *bis*)? Yes, one that is again not referred to by Mr Wiśniewski. (xvii *bis*) is a very approximate formula, but it depends on fourth order terms being negligible as compared with second order terms. The term in question is

$$\frac{1}{576}\frac{hk}{\sigma_x\sigma_y}S\left(\frac{n_{st}}{N}\frac{n_{s+1}-n_{s-1}}{n_s}\frac{n_{t+1}-n_{t-1}}{n_t}\right).$$

Evaluating this term numerically for Mr Wiśniewski's data, we find that it amounts approximately to ·0005, or will only influence the fourth decimal place in the correlation coefficient, while the Sheppard corrections will modify the second decimal place. Accordingly we may consider our results to be sufficiently approximate, if we neglect this fourth order term in this example as suggested on p. 122 of the memoir.

It follows that the reason for Mr Wiśniewski's getting corrections which raise the corrected correlation above unity lies in his overlooking the condition (i) of applying the formula (xvii *bis*), namely, that there must be high contact at both terminals of the marginal totals. When the proper corrections for abruptness are made in the means and standard deviations, then the coefficient of correlation does not exceed unity, but on the contrary is close to the value obtained by the long process.

It may be as well here to show the relation of (xvii) to the closely approximate formula

$$r_{xy}=r_{yC_x}/r_{xC_x} \quad \text{.....(xviii).}$$

* Except in so far as the first set provides another easy proof of (i). We have: Mean value of x for a given $C_x=C_x$ but it also equals since regression is linear $\frac{r_{xC_x}\sigma_x}{\sigma_{C_x}}\times C_x$. Accordingly r_{xC_x} must equal σ_{C_x}/σ_x.

Both formulae follow at once, if we assume linear regressions to hold for all the variables x, y, C_x, C_y. But these not being all linear, my memoir discusses the hypotheses under which they will hold with a very considerable degree of approximation, namely, (i) high contact, (ii) the regressions of x and y linear, and (iii) the product and squares of $\frac{1}{12}h^2/\sigma_x^2$, $\frac{1}{12}k^2/\sigma_y^2$ to be negligible as compared with those quantities themselves. Under these conditions, (xviii) will now be shown to be an approximate as (xvii *bis*).

Starting from (xvii) and using the approximate expressions $\bar{x}_s$ and $\bar{y}_t$ in terms of x_s and y_t on p. 119 of my memoir, we can at once write it in the form

$$
\begin{aligned}
r_{xy}\frac{\sigma_{C_x}}{\sigma_x} = r_{xy}r_{xC_x} &= \frac{S\left(\frac{n_{st}}{N}C_x\bar{y}_t\right)}{\sigma_{C_x}\sigma_y}\frac{\sigma_y^2}{\sigma_{C_y}^2} = \frac{1+\frac{k^2}{12\sigma_y^2}}{\sigma_{C_x}\sigma_y}S\left\{\frac{n_{st}}{N}C_x\left(y_t+\frac{1}{24}k\frac{n_{t+1}-n_{t-1}}{n_t}\right)\right\}\\
&= \left(1+\frac{k^2}{12\sigma_y^2}\right)\left[r_{C_xy}+\frac{1}{24}\frac{k}{\sigma_{C_x}\sigma_y}S\left(\frac{n_{st}}{N}C_x\frac{n_{t+1}-n_{t-1}}{n_t}\right)\right]\\
&= \left(1+\frac{k^2}{12\sigma_y^2}\right)\left[r_{C_xy}+\frac{1}{24}\frac{k}{\sigma_{C_x}\sigma_y}S\left\{\frac{n_{st}}{N}\left(x_s+\frac{n_{s+1}-n_{s-1}}{n_s}\frac{1}{24}h\right)\right\}\right]\frac{n_{t+1}-n_{t-1}}{n_t}\\
&= \left(1+\frac{k^2}{12\sigma_y^2}\right)\left[r_{C_xy}+\frac{1}{24}\frac{k}{\sigma_{C_x}\sigma_y}S\left(\frac{n_{st}x_s}{N}\frac{n_{t+1}-n_{t-1}}{n_t}\right)\right.\\
&\qquad\left.+\frac{1}{576}\frac{hk}{\sigma_{C_x}\sigma_y}S\left(\frac{n_{st}}{N}\frac{n_{s+1}-n_{s-1}}{n_s}\frac{n_{t+1}-n_{t-1}}{n_t}\right)\right].
\end{aligned}
$$

In the second summation we may replace σ_{C_x} by σ_x for $\sigma_{C_x}=\sigma_x\sqrt{1-\frac{1}{12}h^2/\sigma_x^2}$ and the $\frac{1}{12}h^2/\sigma_x^2$ may be neglected in a turn of the fourth order. Accordingly the third summation is negligible, and even in Mr Wiśniewski's case only contributes roughly ·0005 to the right-hand side of the equation.

Now consider the first summation, keeping t constant or sticking to a definite y, $\frac{S(n_{st}x_s)}{n_t}$ = mean x for a given $y_t=\bar{x}_{y_t}$, but if x and y have linear regression, then

$$\bar{x}_{y_t} = r_{xy}\frac{\sigma_x}{\sigma_y}y_t,$$

and the first summation becomes

$$
\begin{aligned}
&-\frac{1}{24}\frac{k}{\sigma_{C_x}\sigma_y}r_{xy}\frac{\sigma_x}{\sigma_y}S\frac{1}{N}y_t(n_{t-1}-n_{t+1})\\
&\quad= -\frac{1}{24}k\sqrt{1+\frac{1}{12}\frac{h^2}{\sigma_x^2}}\frac{1}{\sigma_y^2}r_{xy}\left[S\frac{1}{N}(y_{t-1}n_{t-1}-y_{t+1}n_{t+1})+kS\frac{n_{t-1}}{N}+kS\frac{n_{t+1}}{N}\right].
\end{aligned}
$$

If there be high contact at the terminals, *and in this case only*, the first summation vanishes, and the second and third equal unity. Accordingly, neglecting as before terms of the fourth order of small quantities, the first summation becomes

$$-\frac{1}{12}\frac{k^2}{\sigma_y^2}r_{xy},$$

which is equal to

$$-\frac{r_{xy}}{12}\frac{k^2}{\sigma_y^2}\frac{\sigma_{C_x}}{\sigma_x}\sqrt{1+\frac{1}{12}\frac{k^2}{\sigma_x^2}} = -\frac{r_{xy}}{12}\frac{k^2}{\sigma_y^2}\frac{\sigma_{C_x}}{\sigma_x}$$

to our degree of approximation. Thus

$$r_{xy}\frac{\sigma_{c_x}}{\sigma_x}=\left(1+\frac{k^2}{12\sigma_y{}^2}\right)r_{c_xy}-\frac{k^2}{12\sigma_y{}^2}r_{xy}\frac{\sigma_{c_x}}{\sigma_x},$$

or

$$r_{xy}r_{xc_x}\left(1+\frac{k^2}{12\sigma_y{}^2}\right)=\left(1+\frac{k^2}{12\sigma_y{}^2}\right)r_{yc_x},$$

or, finally,

$$r_{xy}r_{xc_x}=r_{yc_x},$$

which are to be proved. In other words, with our assumptions we get the two relations

$$r_{yc_x}=r_{xy}r_{xc_x} \text{ and } r_{xy}=r_{c_xc_y}/(r_{xc_x}r_{yc_y})$$

as close approximations; these relations are those of linear regression singular partial correlation coefficients.

The conditions of these results holding are stated in my original memoir, the most important one being that of high terminal contacts, which has been entirely overlooked by Mr Wiśniewski. It has long been known that with very high correlations r_{xc_x} and r_{yc_y} may occasionally over correct. But in almost every case where they do so, it will be seen, or it may be suspected, that there is abruptness at some one or other terminal which must be allowed for. We have shown how this may be done in the example selected as an illustration of failure by Mr Wiśniewski. I have, however, to thank him for his paper, because it has enabled me to emphasise again the conditions under which my formulae are approximations, and to indicate those formulae which are absolutely true. It seems to me on re-reading my memoir that I have clearly therein stated these points, but if Mr Wiśniewski has not grasped them, others may likewise fail to do so. I should welcome any alternative means of correction for broad categories when high contact fails, but more than twenty years' experience has shown me that the class index corrections give very reasonable results when high contact limits the frequency distribution.

THE ANGLO-SAXON SKULLS FROM BIDFORD-ON-AVON, WARWICKSHIRE AND BURWELL, CAMBRIDGESHIRE, WITH A COMPARISON OF THEIR PRINCIPAL CHARACTERS AND THOSE OF THE ANGLO-SAXON SKULLS IN LONDON MUSEUMS.

By J. C. BRASH, DORIS LAYARD and MATTHEW YOUNG.

PART I. THE BIDFORD-ON-AVON SKULLS*.

By J. C. Brash and Matthew Young.

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APPENDIX TO PART I.

I. Individual Measurements of the Male Anglo-Saxon Crania from Bidford-on-Avon, with Remarks.

II. Individual Measurements of the Female Anglo-Saxon Crania from Bidford-on-Avon, with Remarks.

III. Individual Measurements of the Male Anglo-Saxon Mandibles from Bidford-on-Avon.

IV. Individual Measurements of the Female Anglo-Saxon Mandibles from Bidford-on-Avon.

* By the courtesy of the Bidford Co-operative Society the skeletal remains are permanently preserved in the Museums of Human and Dental Anatomy in the University of Birmingham.

1. *Introduction.* The bones which are the subject of this communication were first brought to light in June 1921 during the cutting of a new road at Bidford-on-Avon, Warwickshire. In the course of this work a number of skeletons were disinterred along with various articles which indicated the site of an Anglo-Saxon Cemetery. Subsequently it was revealed that the cutting for the road traversed an extensive cemetery, but in 1921 the circumstances were such that no accurate note was preserved of the position of the various graves. The small number of bones available as the result of this initial discovery were merely collected together, many broken in the course of the digging, and, since the relation of the limb and other bones to the skulls was not determined, the exact number of graves opened during 1921 cannot be accurately known.

Arrangements were made, however, for a controlled excavation of the site, which was carried out during the following summers of 1922 and 1923. Details of the methods and course of the excavations, with a discussion of the site, its relation to other Anglo-Saxon cemeteries and of the evidence which suggests that the Bidford Cemetery is to be assigned to the early part of the sixth century, are to be found in two publications in *Archaeologia** by Mr John Humphreys and his collaborators in the work of excavation. The first of these papers includes a map of Bidford showing the exact site of the cemetery which is thus described: "The excavation took place on a site at the back of the High Street, on the north side, in a gravel plateau about 100 yards from the present bank of the river Avon, 150 yards from the Roman Rycknild Street, and 200 yards from the old ford, which lies to the east of the church. The site is on the extreme western border of Warwickshire, within a mile of the Worcestershire boundary." There are also plans of the cemetery, a table giving details of the individual graves and of the grave furniture found with each interment, and three excellent photographs of skeletons and grave furniture *in situ*. The papers in *Archaeologia* are mainly occupied with a detailed and splendidly illustrated description of the archaeological material provided in abundance by the excavations; and we cannot do better, for the purpose of linking the archaeological to the anthropological side of the discovery, than quote the paragraph in which the authors summed up the results of their detailed work.

"The evidence brought to light at Bidford, judging by the size of the graveyard already explored, the number of bodies, and the wealth and variety of the grave furniture, suggests a much earlier settlement in the heart of the Midlands than has generally been admitted, and the great number of cremation burials does not agree with the theory that the proportion of cremation burials decreases the further the settlements advanced up the valleys of the Thames and its tributaries. The materials show so strong a resemblance to those found in the Anglo-Saxon settlements in the Thames valley and its head-waters, that we are forced to conclude that the West Saxons, after penetrating the Cotswolds, descended into

* "An Anglo-Saxon Cemetery at Bidford-on-Avon, Warwickshire," by Humphreys, J., Ryland, J. W., Barnard, E. A. B., Wellstood, F. C., and Barnett, T. G. *Archaeologia*, 1924, LXXIII. 89–116.
"Second Report on the Excavations." *Idem, Ibid.* 1925, LXXIV. 271–288.

the Avon valley, and established themselves on the north bank of the river Avon at Bidford quite early in the sixth century."

"Notes on the Cranial and other Skeletal Characters" have already been published as Appendices to the general accounts in *Archaeologia*, and the purpose of the present paper is to provide data and measurements which may be of use to those who may in the future essay the task of surveying all Anglo-Saxon remains discovered in England and coordinating the scattered evidence of the physical features and interrelations of the various groups in different parts of the country*.

Acknowledgments of support by the Society of Antiquarians, the Birmingham Archaeological Society, and the University of Birmingham, and also to those who personally assisted the work, by giving help at Bidford, by contributions to the Excavation Fund, or by assistance in archaeological matters, have already been made in detail in the papers in *Archaeologia*

2. *Number of Interments.* A perusal of the two papers in *Archaeologia* makes it obvious that the Bidford Cemetery, from the archaeological point of view, must take an important place in the list of Anglo-Saxon discoveries. A preliminary statement of the numbers of skeletons which it has yielded will show that it is equally important in its anthropological aspect. In the table that follows, the probable age and sex distribution of the better-preserved remains are indicated; the number of interments represented by fragmentary bones only and the number of urns found are also stated.

Bidford-on-Avon (1921–22–23).

		Age	♂	♀	Sex?	Totals
Bones	Preserved ...	6	—	—	25	25
		12	—	—	21	21
		20	6	3	3	12
		"*M*"	32	20	20	72
		"*M*+"	16	15	9	40
		"Old"	3	5	2	10
	Fragmentary	"Young"	—	—	12	12
		"Adult"	7	13	10	30
Urns	Well-preserved		—	5	26	31
	Fragmentary		—	—	120+	120
		Totals	64	61	248+	373

"*M*" = Mature.

* Contour tracings of six (6) of the Bidford skulls have already been published by Professor F. G. Parsons, "Anglo-Saxon Skull Contours," *Royal Anthrop. Institute, Occasional Papers*, No. 9, 1928. Tracings of twelve (12) others are available in the Library of the Department of Anatomy, University of Birmingham. All these are indicated in the table of individual measurements. Tables of measurements of the limb bones are also available in the Department of Anatomy, University of Birmingham. [The contour tracings referred to above are of no service for comparison with the long series of contours published in *Biometrika*, as the former are *dioptographic*, and so are not true contours, i.e. plane sections of the skull. Ed.]

The total number of interments uncovered during the excavations of 1922 and 1923, as determined by the discovery of well-preserved skeletons, of fragmentary bones, and of urns, was 229. The grand total of 373 is reached by adding 24 for the number of interments represented by the bones which were collected in 1921*, during the cutting of the road which led to the discovery of the cemetery, and 120 for the minimum number of urns discovered in fragmentary condition throughout the period of the excavations. The number assigned to skeletons discovered in 1921 is also a minimum figure, since no record of the position of each skeleton was made and the bones were mixed; so that it has not been possible to assemble the skeletons with any certainty. The number is in fact derived from the minimum number of individuals determined from the skulls and mandibles alone, and agrees fairly well with the extent of the area excavated and the average spacing of interments determined in the controlled area in 1922—23. The total number of interments excavated in 1922—23, which yielded adult bones in a reasonably fit state for accurate observations and measurements, was about 45.

3. *Sexing of Bones.* The sexing of the bones has presented the usual difficulty, from the purely osteological point of view, of the residuum of doubtful specimens. Where whole skeletons have been preserved the pelvic bones have been taken as the most trustworthy indicators, and the information thus obtained of the range of variation of the other bones in skeletons of practically certain sex has been used to guide the decision where pelvic bones were not available. It has indeed been found that secondary sex characters are very well marked. A further and valuable check has been utilised in the character of the grave furniture where recorded, and this evidence alone has been taken for the purpose of the table in a number of instances. In no instance has there been serious conflict between the archaeological data and the osteological indications; the proportion of the sexes shown in the table may therefore be taken as approximately correct. A number which remain doubtful are indicated as such in the tables, but a sex diagnosis has been supplied for the doubtful skulls of which measurements could be taken.

4. *Age Distribution.* The system that has been adopted is to place the adults in groups according to indications supplied principally by the wearing of the teeth and the state of the cranial sutures, and to group the younger skeletons around the ages 6, 12 and 20, according to the state of the eruption of the teeth and the condition of the epiphysial junctions. Beyond a certain point the determination of age must naturally be approximate only. The eruption of the teeth and the condition of the epiphyses are trustworthy guides up to maturity, but beyond that point only a rough grouping is possible. It is in any case probable that, on the average, age changes would be earlier and more pronounced than in the skeletons of modern people. The largest number of skeletons is found in the "Mature" class—fully adult with moderate wearing of the teeth and still open cranial sutures. The "Mature +" class includes those that show commencing signs of

* The skulls discovered in 1921 are designated by letters in the Appendix Tables; those found in 1922 and 1923 by numbers.

ageing—such as the closure of the cranial sutures associated with increased wear and loss of teeth; and those in which these signs are unequivocal are placed in the "Old" class, which includes a fair proportion of the whole.

The youngest skeleton is that of an infant of about nine months. It is not possible to say what the extreme range of age may have been, but we may assume that most of the "Mature" and the "Mature +" classes would be between 25 and 40. Although the majority fall into these two classes, it is obvious that there is a considerable proportion of young persons and an appreciable number of old.

Apart from all other evidence, the conclusion that is inevitable, from a consideration of the total number of individuals represented and their age and sex distribution, is that we are dealing here with the burial place of a community. But in the absence of evidence of the length of time during which the cemetery was used and of knowledge of the probable death-rate, it is not possible to estimate and hazardous to speculate about the size of the community of which it was undoubtedly the burial place.

5. *The Mode of Burial and the Depth of the Graves.* In most of the burials it was possible to determine the position in which the body had been interred. In the majority the grave had been made sufficiently long to allow the body to be placed in it at full length, but in some the skull was found to be pressed down upon the trunk and the foot bones lay alongside the bones of the leg as if the body had been pressed into a grave too short for it. There appears to have been no exception to the rule that the bodies were buried lying on the back, sometimes with an inclination to the left side. The skull was usually found inclined to the left, though in not a few it was turned to the right. In a considerable number the legs had been crossed at the ankles, generally the right over the left; and in the male burials certainly the position of the arms was fairly constant. The right arm was found to be bent at the elbow, the hand usually folded over the chest, but in some skeletons resting on the right shoulder with the elbow fully flexed. The left arm, in contrast, was usually found extended by the side, the hand in some instances resting on the left thigh or on the lower part of the trunk. The different attitude of the two arms is to be explained by the presence of the spear on the right side, the right arm being naturally folded over it to retain it in position. It is even probable, when the right hand was found to be in the vicinity of the right shoulder, that the spear had actually been placed in the grasp of the hand. (*Archaeologia*, 1924, Figs. 2 and 3.)

The average depth of burial was found to be about 3 ft., the minimum being 2 ft. and the maximum 6 ft.

6. *Orientation of the Graves.* Full details of the orientation of the interments are to be found in the *Archaeologia* publications and in the maps accompanying them. Here it need only be mentioned that some 70 % of the bodies were found with their feet pointing between N.W. and N.E., and that the direction between S. and N.W. appeared to have been carefully avoided. In the part of the cemetery

that is judged to be of later date, there appears to have been a distinct tendency to the E. and W. position; and there is no doubt that this orientation is associated with a comparative lack, and, in some instances, a complete absence of grave furniture.

7. *Grave Furniture.* The spear head, the shield umbo, and the knife were characteristic of the male interments; and the women had been buried with their necklaces of amber and glass beads, and jewelled brooches. Occasionally a small pot was found close to the head of a female skeleton on the right side, and usually there was a small knife on the breast or at the side.

It is an interesting fact that the grave furniture was found to be much less plentiful in the later stages of the excavations; and this is believed to represent the later, possibly post-Christian, portion of the cemetery.

8. *Cinerary Urns.* The association of a considerable number of urn burials with ordinary interments is a feature usual in cemeteries of this date, and of very great interest and significance. It has often been stated that urn burial is an Anglian, inhumation a Saxon, characteristic. The proportion of urns might therefore be held to indicate the degree of Anglian admixture; but in all West Saxon cemeteries that have been investigated, some urn-burials have been found. At Bidford thirty-one more or less perfect cinerary urns were found. Like the interments, they were not arranged in any definite order, but were found scattered about irregularly among the skeletons. The urns were found at an average depth of 2 ft., and the fact that fragments of at least 120 were found, in addition to the smaller number of more or less perfect specimens, is taken to indicate that the urn-burials date from an earlier period than the inhumations. It is conjectured that the majority of the urns were then broken up in making the subsequent graves. The evidence, however, is not quite conclusive that the two methods were not in simultaneous use. Full detailed descriptions and illustrations of the cinerary urns, all of which were hand-made, are to be found in the publications in *Archaeologia*.

9. *Associated Animal Remains.* The majority of the animal bones recovered during the excavations were clearly of modern origin, but it is possible that a few of the bones of pigs, horses, sheep and oxen may be contemporary with the human remains.

10. *The State of Preservation of the Bones.* The condition of the bones is very uneven. Many of them are in excellent condition, though porcelain-like and very brittle. Others are very soft and friable, with flaking and crumbling of the surface. Many of these have had to be treated with balsam varnish to prevent them falling to pieces. Portions of the same skull or long bone are also frequently found to exhibit these extreme conditions of preservation. The excavators of the site attribute these facts to the varying nature of the subsoil. The general good preservation is thought to be due to the excellent drainage provided by the gravel deposit, 20 ft. in thickness, upon which the bodies rested. The accumulation of stagnant water was prevented, and the calcareous nature of the gravels is held to

have neutralised acidity from the atmosphere. Scattered throughout were irregular pockets of boulder clay and it is believed that the softening of many of the skulls and other bones was due to contact with these clay deposits, decalcification having been brought about by the action of sour stagnant water unable to drain away from the surface of the boulder clay.

The number of skulls and other bones available for accurate study, on account of the bad preservation of many of them, consequently falls far below the number of interments. When, in addition to this factor, there is taken into account the number of urn burials, it is found that from a cemetery that is estimated with some degree of accuracy to have contained the remains of at least 373 individuals there are available for the determination of their anthropological characters the skulls and other bones of some 70 individuals only.

The total number of skulls available for measurement, many of them incomplete in different respects, is 51, of which 32 are male and 19 female.

11. *Sex Characters.* There is no doubt that secondary sex characters are very well marked in the skulls. There are very obvious differences in the size of the mastoids and in the muscular impressions in the occipital region; and the superciliary eminences are very extensively developed in a number of male skulls. In a few there is even a filling up of the usual triangular depression between the superciliary eminence and the zygomatic process of the frontal bone. This is noticeably the case in No. 107; and in No. 36, an old male skull, much distorted, there is even the appearance of a moderate torus supraorbitalis.

12. *Age Changes.* There is little of special note to record with regard to the age changes in the skulls. It is probable, though the evidence is by no means decisive, that closure of the sutures commenced at a comparatively early age. The first sites, as a rule, appear to be the usual ones at the lower end of the coronal suture with the pterion region generally, and the hinder part of the sagittal suture, though there are some exceptions. Arthritic changes in the region of the condyles and in the glenoid fossae are almost the rule in those skulls with extensive closure of the sutures.

13. *Remarks on Cranial Anomalies.* (i) *Suture Patterns and Sutural Bones.* The patterns of the sutures in general are of average complexity with the usual scattered sutural (Wormian) bones in the lambdoid suture.

(ii) *Metopism.* There are three examples of definite metopism in the series, Nos. 17, 63, 82. In two of these the suture is closed internally but leaving definite evidence of its presence; and in both of them there are also indications of commencing external union. In both, these signs are distinctly in conformity with the age of the skulls, and union is less advanced than in the other sutures.

(iii) *Metopic Ridge.* Two of the skulls, Nos. A and 106, have slight metopic ridges and in one (D) there is a marked metopic boss occupying the position of the glabella.

(iv) *Pterion.* There is only one example of squamoso-frontal suture in the whole series—on the left side of skull B—in which the junction measures 16 mm. There are several examples of epipteric ossicles; single right in No. 92, joined to the sphenoid: double right in No. C; double left in No. 50 ii; and almost symmetrical single on both sides in No. 37.

(v) *Interparietal.* There are sutural bones of fairly large size in the region of the lambda in a number of skulls; a large one occupying the lambda in No. 93, a child; three symmetrically placed in No. 107; three on the right, one of which is sagittal, in No. D; four in No. 50 i; and a series in No. 82. In only one skull (D) are these ossicles large enough to be interpreted as portions of the interparietal bone. Two large ossicles occupy the upper half of the right portion of the squamous occipital and one of them extends across the middle line, being placed immediately behind the sagittal suture. There are in addition two imperfectly closed sutures in the occipital bone running on each side upwards and towards the middle line from below the asterion. These indicate the separation of the interparietal from the supraoccipital portion of the squamous occipital, and the whole is an example of ossification of the interparietal from several centres.

(vi) *Bregmatic Ossicles.* There are two fine examples of bregmatic ossicle (os antiepilepticum). In No. 5, a male, the ossicle lies entirely behind the coronal suture, measures 27 mm. in antero-posterior length by 15 mm. in maximum breadth, and is slightly constricted in the middle. In No. 89, female, the ossicle indents the line of the coronal suture but lies almost entirely behind it. It measures 19 mm. in antero-posterior length by 14 mm. in breadth. In neither of these examples is there any sign of the union of the ossicle. In No. 82, a metopic skull, the interfrontal suture joins the coronal to the right of the sagittal and there is evidence of the former presence of an anterior bregmatic ossicle which has united to the left half of the frontal bone. The association of metopism with the development of a bregmatic ossicle is noteworthy.

(vii) *Asterionic Ossicles.* There are also two examples of large asterionic ossicles, and it is probably more than a coincidence that these occur in the same two skulls as the bregmatic ossicles—Nos. 5 and 89. They are both on the left side and show signs of uniting with the surrounding bones.

(viii) *Obelionic Depression.* There are five examples of a groove along the sagittal suture in the region of the obelion. In three of these (Nos. 40, 75 i, and 78) the groove is well marked, in one of them (No. 75 i) it extends to within 2 cm. of the bregma and is also continued on the occipital bone as a slight depression from lambda to inion. The other two examples (Nos. 46 and 66) are slight, and in No. 46 it is associated with symmetrical depressions extending on each side along the lambdoid suture. One skull (No. C) exhibits a fairly obvious sagittal crest.

(ix) *Post-coronal Sulcus.* There is no definite example of this in the whole series.

(x) *Coronal Ossicle.* There is one example of ossicle in the coronal suture, on the right side just below the temporal line; this occurs in No. 82, already noted as a metopic skull with an anterior bregmatic ossicle.

(xi) *Carotico-Clinoid Foramen.* In No. 37, owing to damage to the skull, it may be observed that there is a complete carotico-clinoid foramen on both sides. It may be noted that this is associated with symmetrical epipteric ossicles.

(xii) *Paroccipital and Paramastoid Processes.* There is a paroccipital process in No. C and a nipple-like paramastoid process in No. D, both on the right side.

(xiii) *Ossified Trochlea.* The trochlea is ossified in both orbits of No. 60, a skull which does not show any definite signs of ageing.

(xiv) *Divided Malar Bones.* There is no example of divided malar in the series, but it may be noted that both the malars are transversely grooved in No. A.

(xv) *Nasal Aperture.* Some notes on the form of the lower border of the anterior (pyriform) aperture of the nose will be found in the remarks appended to the Table of Measurements. It is worthy of mention here that a complete sharp limiting margin is uncommon in these skulls, that prenasal fossae are present in several, and that in one (No. E) there are infranasal grooves (prenasal sulci) of anthropoid type.

(xvi) *Asymmetry.* As is the case in any collection of skulls, there is a considerable amount of asymmetry to be observed, especially in the region of the base. It is probable that the immediate causes of asymmetry are to be found not only in the well-known asymmetrical development of the occipital lobes of the cerebrum, but also in individual habits in the "wear of the head" and in the use of the jaws. The correlation of the different points in which a skull may be asymmetrical would be a very interesting study; one or two points as shown in this series may be indicated. The occipital region has been noted as obviously asymmetrical in 7 skulls; the left is more projecting in 6 of these, and the right in only 1. In 5 of the 6 skulls with left occipital prominence it is also noted that the right mastoid process is larger than the left. Taking the skulls as a whole the mastoids are noted as asymmetrical in 18, and of these the right is larger in 10, the left in 8. The occipital condyles are noted as asymmetrical in 6 skulls, the right condyle being anterior to the left in 2, the left anterior to the right in 4. There is, however, no obvious association of position of the condyle with either occipital prominence or larger mastoid. The glenoid cavities have been noted as asymmetrical in 13 skulls and of these the right is deeper than the left in 5, the left deeper than the right in 8. Again there is no obvious relation between these facts and the other asymmetrical points noticed. The last point in which asymmetry has been noted is in the size of the nasal bones; this is definite in 8 skulls and in all but one of these the right bone is the larger and passes over the median plane at its frontal articulation. It may be surmised that this condition is correlated with forward projection of the frontal bone occurring in skulls with a well-marked left occipital projection; but as a matter of fact in this series asymmetry of the frontal bone itself is not at all obvious, and the asymmetrical nasal bones occur,

with two exceptions, in skulls which do not show left occipital projection, and these two exceptions include the only example of a larger left nasal bone.

(xvii) *Irregularity and Malocclusion of the Teeth.* It should be recorded here that the Bidford series has been the subject of a special investigation of the incidence of irregularities of the teeth and of malocclusion. These conditions, in mild or severe degree, have been found in a surprising number of the skulls; details and discussion will be found in "The Aetiology of Irregularity and Malocclusion of the Teeth" (Dental Board of the U.K. Lectures, J. C. Brash, 1929) and "Some Notes on the Dentitions of Anglo-Saxon Skulls from Bidford-on-Avon with special reference to Malocclusion" (*Trans. B.S.S.O.*, K. C. Smyth, 1933).

14. *Brief Description and Discussion of the Principal Cranial Characters.* The main object of Part I of this paper is to place on record accurate data concerning the cranial and mandibular characters of these Bidford Saxons. The principal measurements of the individual skulls that were sufficiently well preserved for taking at least a number of the measurements included in the scheme of measurement usually followed at the Biometric Laboratory, University College, London, and in accordance with the technique practised there are tabulated in the Appendix with brief notes on any distinctive or peculiar features of particular skulls that seem worthy of mention.

Tables of the average values of these characters for the separate sexes with the numbers of observations on which the averages are based are given in the text on pp. 390—391. As the numbers of skulls on which the means are based rarely exceed and are usually considerably below 20, the standard deviations—the measures of variability—of the characters have not been considered sufficiently representative or reliable for tabulation.

A selected series of the cranial characters will be considered individually and in the aggregate later, when a comparison is made between them and the corresponding characters in the groups of Anglo-Saxon skulls from Burwell in Cambridgeshire and from the London Museums, but some of the more general features of the skulls as a group will be indicated here by reference to a few of the principal cranial characters.

(i) *Cranial Capacity.* In only 11 of the skulls, 6 male and 5 female, did the condition of the specimens warrant the risk of estimating the cubic capacity by the direct method, No. 8 shot having been used as a medium. These are entered in the table of measurements of individual skulls. For the 6 male skulls the average cubic capacity was 1553 c.c. and for the 5 female skulls 1378 c.c. The mean capacities for male and female skulls in the series were also estimated from the product of the serial means of the three dimensions, greatest length (L), maximum breadth (B) and basio-bregmatic height (H') by using Miss Hooke's formulae*, namely:

$$\text{♂}\quad C \text{ (in c.c.)} = \cdot 000{,}366\, LBH' + 198{\cdot}87,$$

$$\text{♀}\quad C \text{ (in c.c.)} = \cdot 000{,}366\, LBH' + 199{\cdot}43.$$

* *Biometrika*, Vol. XVIII. (1926), p. 33.

The values thus obtained, 1595·7 for the male and 1411 c.c. for the female, are rather in excess of the mean capacities obtained from the smaller numbers of specimens by the direct method*.

(ii) *Cephalic Index* (100 B/L). Omitting one or two skulls which were obviously distorted, the cephalic indices were obtained for 29 skulls, of which 10 were female. In the males the index ranges from 69·5 to 77·8 with a mean value of 73·5; in the females the range is from 68·4 to 76·9 with a mean of 73·8. The distribution of the skulls according to type is as follows:

Class	Range	Numbers			Per cent.
		♂	♀	♂ and ♀	♂ and ♀
Dolichocephalic	−75	13	6	19	65·5
Mesaticephalic	75—77·4	3	4	7	24·1
	77·5—79·9	3	0	3	10·3
Brachycephalic	80+	0	0	0	0

Nineteen of the skulls, 66 %, fall below the 75 mark and are therefore to be reckoned definitely dolichocephalic; the remainder fall within the mesaticephalic class, not a single brachycephalic skull being included in the series. It is also of interest to note that on a division of the mesaticephalic class into two subclasses at 77·5, as suggested by Sir William Turner, in order to indicate the affinities in the intermediate group, the proportion that may be considered to show a distinctly long-headed tendency is almost 90 % of the total number. We are therefore clearly dealing with a sample of a strongly dolichocephalic community.

Length-height Index (100 H'/L). The classification of the skulls according to this index is as follows:

Class	Range	Numbers			Per cent.
		♂	♀	♂ and ♀	♂ and ♀
Chamaecephalic	−70	0	2	2	11·1
Orthocephalic	70—75	10	2	12	66·7
Hypsicephalic	75+	1	3	4	22·2

* [It should be noted here that Miss Hooke's formulae are based on the Biometric Laboratory method, which involves the use of *mustard* seed. The results obtained by the use of shot are not comparable with those obtained by mustard seed, witness the exaggerated values obtained by Broca's use of shot. Nor is it legitimate to put into the product term in Miss Hooke's formulae the product of the serial means, for the product of the means is not the mean product. Ed.]

From this point of view the skulls are predominantly of the orthocephalic or intermediate type. In the males the length-height index ranges from 70·0 to 76·2 with a mean value of 71·8, and in the females from 66·8 to 76·9 with a mean value of 72·4.

Height-breadth Index ($100B/H'$). For this index we have the following distribution:

Class	Range	Numbers			Per cent.
		♂	♀	♂ and ♀	♂ and ♀
Hypsistenocephalic	−100	3	2	5	29·4
Platychamaecephalic	100+	8	4	12	70·6

The majority thus fall into the platychamaecephalic class. In the males the index ranges from 98·5 to 109·3 with a mean value of 102·2; in the females the range is from 96·3 to 112·2 with a mean value of 103·5.

The impression conveyed from a study of the foregoing cranial indices is that we are dealing with skulls in which on the whole a low mean cephalic index, i.e. dolichocephaly, is associated with a definite preponderance of breadth over height.

It may be recalled here that this combination of features was pointed out by Sir Wm Turner as characteristic of his series of Scottish skulls*. He then remarked: "A striking feature of the Scottish crania, therefore, was the preponderance of the cephalic index over the vertical index, notwithstanding the considerable number of dolichocephalic skulls in the series, and in this respect the crania favoured the brachycephalic rather than the dolichocephalic type. The Scottish skulls are platychamaecephalic."

Upper Facial Index ($100\,G'H/GB$). According to the value of this index, the skulls may be distributed as follows:

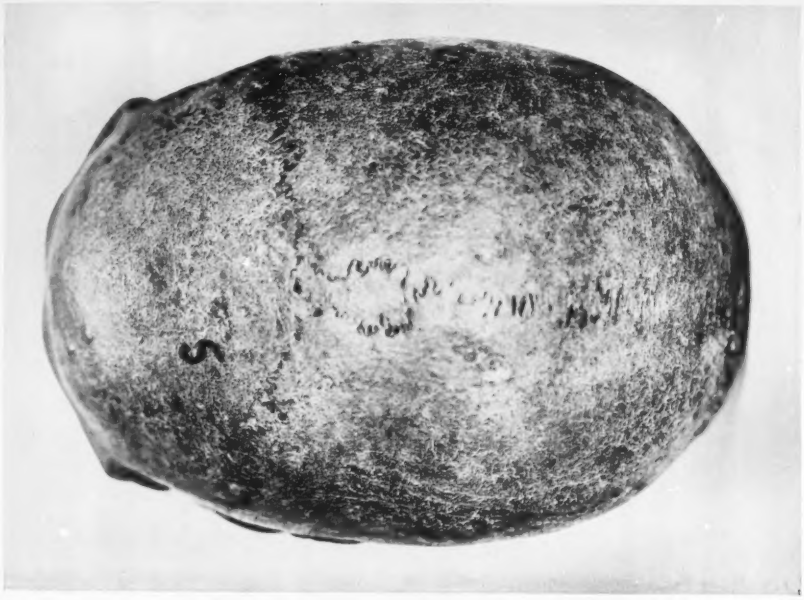
Class	Range	Numbers			Per cent.
		♂	♀	♂ and ♀	♂ and ♀
Chamaeprosopic	−75	12	7	19	76·0
Leptoprosopic	75+	5	1	6	24·0

The majority of the specimens thus show a relatively broad face. In the males the index varies from 65·3 to 80·3 with a mean value of 72·4; in the females the range is from 64·4 to 77·1 with an average of 71·5.

* *Trans. Roy. Soc. Edin.*, Vol. XL. Part III.

Normae laterales of Male Anglo-Saxon Skulls from Bidford-on-Avon.

Normae faciales of Anglo-Saxon Skulls from Bidford-on-Avon.



CM
No. 5

CM
No. A

Normae verticales of Anglo-Saxon Skulls from Bidford-on-Avon.

UNIV. OF MICH.

Normae occipitales of Anglo-Saxon Skulls from Bidford-on-Avon.

UNIV. OF MICH.

Prognathism. Inspection of the skulls after orientation in the Frankfort plane conveyed the definite impression that some of the females, if not the majority, were relatively prognathic as compared with the males. As the mean value of the profile angle ($\angle P$) in the males in which this measurement was taken was 85·2°, whereas in the females it was 83·1°, or 2 degrees less, there would appear to be a reasonable basis for the belief. Further reference will be made to this feature later (pp. 395—396).

Cranial Types in the Series. In the brief notes on the cranial characters appended to the first publication in *Archaeologia*, it was suggested that the distinction to be drawn between the definitely dolichocephalic skulls and those with a mesaticephalic index was associated with an obvious and striking difference in the appearance of the skulls in *norma verticalis*, and it was there stated that "there are found to be marked differences in the amount of parietal bulge, so striking in the well-characterised males as to lead to an initial distinction of two types best described as 'ovoid' and 'ellipsoid'." With the greater number of skulls now available, it is clear that while the sharpness of this distinction has been modified by the addition of intermediate forms, yet it is a distinction which stands confirmed. The two skulls, Nos. A and 5, which have been chosen for reproduction in four normae, are perhaps the best examples in the series of these two types which have been designated "ovoid" and "ellipsoid," the ovoid being relatively broad and low, and the ellipsoid relatively high and narrow. While this differentiation into two types is mainly based on their relatively great difference in parietal breadth, which can be clearly observed not only in *norma verticalis* but also in the corresponding facial and occipital normae, the two types differ in other features notably in the characters of the facial skeleton. Though there seems to be, from the cranioscopic point of view, a reasonable basis for the establishment of a valid distinction between types within the series, it should be noted that the variabilities shown by the characters L ($\sigma = 5{\cdot}43$), B ($\sigma = 5{\cdot}97$), $100\,B/L$ ($\sigma = 2{\cdot}74$) and $100\,GH'/GB$ ($\sigma = 4{\cdot}84$) in the male group are not greater than those shown in the corresponding characters in the London Museums series of Anglo-Saxons or the Whitechapel seventeenth-century Londoners, both of which have been considered fairly homogeneous groups of skulls.

The larger question of the bearing of these observations upon the problem of the origin of these Bidford-on-Avon people can be adequately discussed only by someone who may review together all the recorded data on Anglo-Saxon remains.

Mandibles. After a careful preliminary survey of the mandibles that had been excavated, it was concluded that the total number of specimens represented in whole or in part was probably about 86. Some of these were not associated with crania. 37 of this total were fairly complete and adult or aged. Of these 20 were identified as male, 13 as female, and 4 of aged individuals were considered of doubtful sex. 17 of the total were less complete. Of these 10 were male, 6 female and in 1 the sex was doubtful. 23 adult specimens were only represented by fragments, and 9 more or less complete were obviously juvenile.

When it was decided to take detailed measurements of the adult mandibles, 50 specimens, 31 of which were identified as ♂ and 19 as ♀, were found to be sufficiently complete to provide a number at least, if not the majority, of the usual measurements.

These mandibles were measured in accordance with the scheme of measurement now followed at the Biometric Laboratory, University College, London. As compared with the original scheme which is described in detail in *Biometrika*, Vol. XIV. pp. 253—260, the number of characters of which measurements are taken is considerably reduced. We are indebted to Dr G. M. Morant for the abbreviated list of measurements now usually taken. The measurements of the individual mandibles are given in detail in the Appendix (Tables III and IV). The mean values of the characters measured and the indices computed from these with the numbers of observations on which the averages are based are shown for males in Table VI and for females in Table VII. A detailed comparison of the mandibular characters individually and in the aggregate with the corresponding characters in the two series of Anglo-Saxon mandibles from Burwell and from the London Museums will be given later (pp. 398—404).

Serial Letter or No.	Age	C	F	L	B	B'	H'	OH	LB	Q'	S	S_1	S_2	S_3	S_1'	S_2'	S_3'
A^*	M	1645	195·0	197·0	153·0	98·0	140·0	117·0	110·0	317·5	382·0	130·0	129·0	123·0	117·0	115·0	96·5
B^*	M	1600	196·0	199·0	145·0	99·0	140·0	118·5	106·0	321·0	394·0	133·0	139·0	122·0	118·5	125·5	98·0
C^*	M	1665	198·5	202·0	147·0	104·0	147·0	124·5	111·0	315·0	395·5	136·5	142·5	116·5	121·5	129·0	95·5
D^*	M	1355	189·0	191·0	137·0	96·0	133·0	107·0	107·0	292·5	354·0	129·0	110·0	115·0	112·0	101·5	93·0
F	$M+$	—	187·5	190·0	147·0	99·5	—	—	—	320·0	388·0	131·0	129·0	128·0	114·5	116·0	102·0
H	20	—	—	—	—	—	—	—	—	—	—	—	129·0	—	114·0	118·5	—
J	$M+$	—	—	—	—	—	—	—	—	—	—	—	116·0	121·5	—	110·0	97·0
2 a	M	—	—	188·0	146·0	—	—	—	101·5	—	—	—	—	124·0	—	—	104·0
5	$M+$	1469	190·0	192·0	135·0	93·0	137·0	115·5	102·0	297·0	386·0	129·0	138·5	118·5	110·0	122·0	99·0
6 a	$M+$	—	—	—	—	95·5	—	—	—	—	—	132·0	—	—	112·5	—	—
17	$M+$	—	—	—	—	93·0	—	110·0	—	—	—	120·0	—	—	104·0	—	—
24	M	—	189·0	191·0	144·0	—	—	—	—	310·0	394·0	135·0	135·0	124·0	116·5	119·5	96·0
36	A	—	180·5	184·0	—	90·0	—	—	—	—	358·0	115·0	130·0	113·0	102·5	116·0	95·0
37	M	—	—	—	—	93·0	—	110·0	—	293·0	—	122·0	—	—	106·0	—	—
48	M	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
50 i	M	—	—	—	—	96·0	—	—	—	303·0	—	125·0	145·0	—	102·5	123·0	—
60	M	—	—	—	—	97·0	—	125·0	—	320·0	—	133·0	130·0	—	115·0	115·5	—
66	A	—	192·0	193·5	137·5	100·5	135·5	—	107·0	305·0	384·0	134·0	129·0	121·0	118·5	116·5	96·0
70	20	—	—	—	—	95·0	—	—	—	—	—	115·0	—	—	104·5	—	—
78	A	—	—	—	—	—	—	—	—	—	—	124·0	124·0	—	110·0	115·0	—
82	$M+$	—	193·0	193·0	136·5	106·0	—	—	—	302·0	380·0	127·0	126·5	126·5	111·0	115·0	101·0
84	M	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
92*	M	—	187·0	189·0	147·0	100·0	144·0	—	100·0	320·0	—	140·0	135·0	—	122·0	119·0	—
106	$M+$	—	191·5	194·0	119·5	91·0	140·0	—	104·0	297·0	390·0	129·0	131·0	130·0	113·5	118·5	705·5
107	M	1584	193·5	199·0	143·0	91·0	140·0	113·0	110·0	297·0	383·0	122·0	140·0	121·0	111·0	126·0	97·0
112 a	M	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
118	$M+$	—	195·0	197·0	137·0	96·4	—	—	—	—	—	—	—	—	—	112·5	109·5
122	M	—	197·3	198·0	147·0	94·5	—	—	—	—	—	—	—	—	122·0	116·0	94·2
147 i	$M+$	—	191·0	192·8	146·2	95·4	139·2	—	103·0	—	—	—	—	—	119·0	117·0	98·0
157	M	—	180·0	182·0	130·0	95·5	—	—	—	—	—	—	—	—	111·5	111·3	113·3
169	$M+$	—	193·0	195·0	140·5	102·5	141·0	—	113·0	—	—	—	—	—	110·3	114·9	105·1
183	20	—	177·5	179·0	133·5	96·9	—	—	—	—	—	—	—	—	109·9	116·0	—

Serial Letter or No.	Age	C	F	L	B	B'	H'	OH	LB	Q'	S	S_1	S_2	S_3	S_1'	S_2'	S_3'
E	M	1410	186·5	187·0	142·0	96·0	132·0	114·5	96·0	307·0	382·5	126·0	130·5	126·0	108·5	118·5	103·5
G	A	—	—	—	—	—	—	—	—	—	—	121·0	—	—	108·5	—	—
13	$M+$	—	179·0	180·0	137·5	92·0	—	111·0	—	—	—	125·0	130·0	—	111·0	112·5	—
27	M	—	196·0	196·0	139·0	97·5	—	—	—	308·0	—	129·0	132·0	—	111·0	121·0	—
40	A	—	—	—	—	—	—	—	—	—	—	—	136·0	—	—	123·0	—
46	A	1425	190·0	191·0	143·0	98·5	127·5	112·0	102·5	294·0	370·0	116·0	125·0	129·0	105·5	114·0	101·0
56 i	M	1288	175·0	175·0	129·5	93·0	134·5	104·5	101·5	—	359·0	126·0	125·0	108·0	109·0	110·5	92·5
56 ii	M	1399	187·5	188·0	139·0	95·5	144·0	114·0	107·0	301·0	376·0	125·0	131·0	120·0	110·0	118·5	103·5
63	$M+$	—	192·0	194·0	136·0	101·0	—	—	—	320·0	—	129·0	120·0	—	111·5	110·5	—
75 i	M	—	180·0	180·5	—	92·0	—	—	—	—	—	118·0	121·0	—	104·0	109·5	—
79	$M+$	—	—	—	—	86·5	—	—	—	—	—	126·0	—	—	111·0	—	—
89	20	1368	176·0	179·5	138·0	95·0	128·0	—	98·5	290·0	348·0	111·0	122·0	115·0	103·5	109·5	99·0
90	M	—	—	189·0	—	95·0	130·0	—	99·0	—	—	124·0	—	—	110·0	—	—
121	M	—	190·5	193·0	132·0	105·3	—	—	—	—	—	—	—	—	105·3	123·0	—
152*	M	—	185·0	187·0	139·0	97·5	129·0	107·5	106·0	300·0	357·0	120·0	120·0	117·0	110·0	107·0	97·0
178 i	M	—	166·3	167·1	127·5	91·2	127·0	—	93·5	—	—	—	—	—	99·5	109·4	90·3
181 i	$M+$	—	—	—	—	96·0	—	—	—	—	—	—	—	—	108·5	109·8	—
187	$M+$	—	—	184·2	183·0	—	—	—	—	—	—	—	—	—	115·0	117·3	—
190 i	$M+$	—	—	173·0	172·6	—	97·2	—	—	—	—	—	—	—	110·0	117·0	—

APPENDIX I. INDIVIDUAL MEASUREMEN

S_1'	S_2'	S_3'	U	GH	$G'H$	GB	J	NH'	NB	DC	O_1, R	O_1, L	O_2, R	O_2, L	EOW
117·0	115·0	96·5	550·0	121·0	75·5	94·0	143·0	57·0	25·2	20·0	43·0	43·0	35·0	36·3	104·0
118·5	125·5	98·0	547·0	—	66·0	101·0	139·0	50·5	24·0	24·5	—	39·5	—	30·5	101·5
121·5	129·0	95·5	547·0	—	77·5	97·5	139·0	54·2	26·0	24·0	40·5	39·7	36·0	33·7	103·0
112·0	101·5	93·0	525·0	125·0	72·0	102·0	134·5	54·5	22·0	14·5	40·5	43·5	35·5	36·0	100·5
114·5	116·0	102·0	—	—	—	—	—	—	—	—	—	—	—	—	103·0
114·0	118·5	—	—	—	68·5	—	—	50·0	21·0	—	—	38·0	—	—	—
—	110·0	97·0	—	—	—	—	—	—	—	—	—	—	—	—	—
—	—	104·0	—	113·0	64·0	91·5	136·0	49·0	24·0	23·0	—	40·5	—	33·0	101·0
110·0	122·0	99·0	529·0	116·5	69·5	91·0	142·0	55·5	24·0	23·5	40·5	42·0	35·5	36·0	103·0
112·5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	96·0
104·0	—	—	—	—	67·0	91·0	—	51·0	22·0	21·0	39·5	39·0	33·5	35·0	93·5
116·5	119·5	96·0	544·0	—	—	—	—	—	—	—	—	—	—	—	—
102·5	116·0	95·0	—	—	—	—	—	—	—	21·0	—	37·0	—	35·0	—
106·0	—	—	—	—	64·0	89·5	126·0	48·0	22·0	20·5	39·5	39·5	35·5	36·0	94·5
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
102·5	123·0	—	—	—	—	—	—	—	—	24·5	36·0	—	29·0	—	97·0
115·0	115·5	—	—	118·0	68·5	97·0	131·0	53·5	22·0	22·0	41·0	41·5	34·5	33·0	96·5
118·5	116·5	96·0	534·0	—	72·0	99·0	138·0	53·0	—	22·0	—	40·5	—	34·0	102·5
104·5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
110·0	115·0	—	—	—	67·0	99·0	—	51·5	24·5	—	—	38·0	—	34·5	—
111·0	115·0	101·0	535·0	130·0	80·0	—	—	—	—	29·0	—	38·0	—	37·0	104·0
—	—	—	—	—	76·0	—	—	56·0	24·0	—	—	—	—	—	—
122·0	119·0	—	535·0	124·0	69·5	98·0	136·0	50·0	24·0	20·5	41·5	41·0	35·0	34·0	101·5
113·5	118·5	105·5	529·0	110·5	64·5	—	—	50·5	26·0	21·0	41·0	—	33·0	—	97·0
111·0	126·0	97·0	533·0	116·5	69·0	100·0	137·0	51·0	26·0	24·0	40·0	40·5	30·0	30·0	101·5
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
—	112·5	109·5	—	—	—	—	—	—	—	—	—	—	—	—	—
122·0	116·0	94·2	—	—	—	—	—	—	—	—	—	—	—	—	—
119·0	117·0	98·0	—	117·0	73·6	93·0	—	52·9	25·3	21·0	—	42·6	—	35·3	—
111·5	111·3	113·3	—	—	63·3	91·9	—	45·7	26·8	21·0	41·0	39·7	32·5	33·5	—
110·3	114·9	105·1	—	121·0	74·7	94·9	130·0	54·7	23·9	26·5	42·0	43·3	33·0	34·2	—
109·9	116·0	—	—	103·0	58·3	89·0	—	46·8	25·6	24·5	39·0	41·3	30·0	30·0	—

* Dioptographic tracin

APPENDIX II. INDIVIDUAL MEASUREMEN

S_1'	S_2'	S_3'	U	GH	$G'H$	GB	J	NH'	NB	DC	O_1, R	O_1, L	O_2, R	O_2, L	EOW
108·5	118·5	103·5	—	—	—	—	—	—	—	—	—	—	—	—	93·5
108·5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
111·0	112·5	—	—	119·5	—	—	—	—	—	20·0	38·5	—	32·5	—	92·0
111·0	121·0	—	531·0	110·0	65·0	93·5	—	50·0	22·0	23·0	41·0	39·0	34·5	34·5	98·0
—	123·0	—	—	—	—	—	—	—	—	—	—	—	—	—	—
105·5	114·0	101·0	534·0	115·0	69·0	95·0	134·0	51·0	23·5	20·5	43·0	41·0	38·5	36·5	101·5
109·0	110·5	92·5	486·0	96·0	58·0	90·0	—	43·5	23·5	18·5	37·0	37·0	31·0	32·0	90·5
110·0	118·5	103·5	528·0	117·0	70·0	93·5	—	51·0	27·5	18·0	41·0	41·0	33·5	34·5	98·0
111·5	110·5	—	534·0	117·5	69·5	94·0	—	50·5	23·5	24·0	—	39·0	—	34·5	97·0
104·0	109·5	—	—	—	—	—	—	—	—	—	—	—	—	—	—
111·0	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
103·5	109·5	99·0	495·0	101·5	64·0	83·0	120·0	50·5	24·5	21·0	39·0	—	34·5	—	93·5
110·0	—	—	—	116·0	68·0	—	—	50·0	23·0	23·0	—	39·0	—	35·0	96·0
105·3	123·0	—	—	—	—	—	—	—	—	—	—	—	—	—	—
110·0	107·0	97·0	515·0	111·5	67·0	91·0	129·0	49·0	24·0	—	42·5	42·0	34·0	35·0	—
99·5	109·4	90·3	—	100·0	61·0	87·5	121·5	45·9	23·5	24·0	39·5	39·5	28·0	29·6	—
108·5	109·8	—	—	—	—	—	—	—	—	—	—	—	—	—	—
115·0	117·3	—	—	—	—	—	—	—	—	—	—	—	—	—	—
110·0	117·0	—	—	—	63·7	91·5	—	48·5	23·0	22·0	43·5	43·2	35·5	34·7	—

* The measurements of this skull were not used in calculating t

ASUREMENTS OF MALE ANGLO-SAXON CRANIA FROM BIDFORD-ON-AVON.

	O_2, L	*EOW*	G_1	G_2	*GL*	*fml*	*fmb*	100 *B/L*	100 *B/F*	100 *H'/L*	100 *B/H'*	100 *G'H/GB*	100 *G'H/J*	100 *NB/NH'*	100 $O_2/O_1, R$	100 $O_2/O_1, L$	100 G_2/G_1	100 *fmb/fml*	*Oc. I*	∠*P*
	36·3	104·0	45·5	45·0	98·0	43·0	37·5	77·7	78·5	71·1	109·3	80·3	52·8	44·2	81·4	84·4	98·9	87·2	56·0	88·0
	30·5	101·5	48·0	42·0	105·0	40·0	30·0	72·9	74·0	70·4	103·6	65·3	47·5	47·5	—	77·2	87·5	75·0	57·3	83·0
	33·7	103·0	44·0	39·5	100·5	40·0	33·0	72·8	74·1	72·8	100·0	79·5	55·8	48·0	88·9	84·9	89·8	82·5	58·6	89·0
	36·0	100·5	47·5	42·5	104·0	46·0	35·0	71·7	72·5	69·6	103·0	70·6	53·5	40·4	87·7	82·8	89·5	76·1	57·7	88·0
	—	103·0	—	—	—	—	—	77·4	78·4	—	—	—	—	—	—	—	—	—	56·8	—
	—	—	—	—	—	—	—	—	—	—	—	—	—	42·0	—	—	—	—	—	—
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	56·9	—
	33·0	101·0	47·5	39·5	101·5	33·5	31·5	77·7	—	—	—	69·9	47·1	49·0	—	81·5	83·2	94·0	60·6	83·0
	36·0	103·0	51·0	39·0	97·5	41·0	34·0	70·3	71·1	71·4	98·5	76·4	48·9	43·2	87·7	85·7	76·5	82·9	60·2	83·0
	—	96·0	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
	35·0	93·5	48·0	37·5	—	—	—	—	—	—	—	73·6	—	43·1	84·8	89·7	78·1	—	—	84·0
	—	—	—	—	—	35·0	32·0	75·4	76·2	—	—	—	—	—	—	—	—	91·4	55·5	—
	35·0	—	—	—	—	—	—	—	—	—	—	—	—	—	—	94·6	—	—	60·9	—
	36·0	94·5	47·0	34·5	—	—	—	—	—	—	—	71·5	50·8	45·8	89·9	91·1	73·4	—	—	84·0
	—	—	47·0	37·0	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
	—	97·0	—	—	—	—	—	—	—	—	—	—	—	—	80·6	—	—	—	—	—
	33·0	96·5	46·5	35·0	—	—	—	—	—	—	—	70·6	52·3	41·1	84·1	79·5	75·3	—	—	87·0
	34·0	102·5	50·5	38·0	102·5	36·0	30·0	71·1	71·6	70·0	101·5	72·7	52·2	—	—	84·0	75·2	83·3	56·6	—
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
	34·5	—	47·0	37·0	—	—	—	—	—	—	—	67·7	—	47·6	—	90·8	78·7	—	—	—
	37·0	104·0	49·0	—	—	—	—	70·7	70·7	—	—	—	—	—	—	97·4	—	—	57·0	—
	—	—	49·5	44·5	—	—	—	—	—	—	—	—	—	42·9	—	—	89·9	—	—	—
	34·0	101·5	—	39·5	100·0	—	—	77·8	78·6	76·2	102·1	70·9	51·1	48·0	84·3	82·9	—	—	—	—
	—	97·0	47·0	43·0	—	38·5	29·5	71·9	72·8	72·2	99·6	—	—	51·5	80·5	—	91·5	76·6	58·0	—
	30·0	101·5	50·5	36·0	106·5	36·0	31·0	71·9	73·9	70·4	102·1	69·0	50·4	51·0	75·0	74·1	71·3	86·1	57·2	83·0
	—	—	44·0	37·0	—	—	—	—	—	—	—	—	—	—	—	—	84·1	—	—	—
	—	—	—	—	—	—	—	69·5	70·3	—	—	—	—	—	—	—	—	—	—	—
	—	—	—	—	—	—	—	74·2	74·5	—	—	—	—	—	—	—	—	—	—	—
	35·3	—	47·7	37·2	103·4	41·6	31·2	75·8	76·5	72·2	105·0	79·1	—	47·8	—	82·9	78·0	75·0	—	—
	33·5	—	45·0	36·8	—	—	—	71·4	72·2	—	—	68·9	—	58·6	79·3	84·4	81·8	—	—	—
	34·2	—	—	—	101·8	37·7	30·1	72·1	72·8	72·3	99·6	78·7	57·5	43·7	78·6	79·0	78·6	79·8	—	—
	30·0	—	—	—	—	—	—	74·6	75·2	—	—	65·5	—	54·7	76·9	72·6	—	—	—	—

ptographic tracings of the contours of these skulls have been published by Parsons.

SUREMENTS OF FEMALE ANGLO-SAXON CRANIA FROM BIDFORD-ON-AVON.

	O_2, L	*EOW*	G_1	G_2	*GL*	*fml*	*fmb*	100 *B/L*	100 *B/F*	100 *H'/L*	100 *B/H'*	100 *G'H/GB*	100 *G'H/J*	100 *NB/NH'*	100 $O_2/O_1, R$	100 $O_2/O_1, L$	100 G_2/G_1	100 *fmb/fml*	*Oc. I*	∠*P*
	—	93·5	42·0	40·5	—	—	—	75·9	76·1	70·6	107·6	—	—	—	—	—	96·4	—	58·8	87·0
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
	—	92·0	—	—	—	—	—	76·4	76·8	—	—	—	—	—	84·4	—	—	—	—	83·5
	34·5	98·0	46·0	40·5	—	—	—	—	—	—	—	69·5	—	44·0	84·1	88·5	88·0	—	—	—
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
	36·5	101·5	46·0	37·0	99·0	—	—	74·9	75·3	66·8	112·2	72·6	51·5	46·1	89·5	89·0	80·4	—	56·0	85·0
	32·0	90·5	45·0	37·5	98·0	32·5	26·5	74·0	74·0	76·9	96·3	64·4	—	54·0	83·8	86·5	83·3	81·5	62·9	80·0
	34·5	98·0	49·5	41·5	101·0	40·0	30·0	73·9	74·1	76·6	96·5	74·9	—	53·9	81·7	84·1	83·8	75·0	63·9	80·0
	34·5	97·0	47·5	35·5	—	—	—	70·1	70·8	—	—	73·9	—	46·5	—	88·5	74·7	—	—	—
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
	—	93·5	42·0	32·0	91·0	38·0	30·0	76·9	78·4	71·3	107·8	77·1	53·3	48·5	88·5	—	76·2	78·9	63·5	—
	35·0	96·0	44·0	38·5	91·5	—	—	—	—	68·8	—	—	—	46·0	—	89·7	87·5	—	—	—
	—	—	—	40·7	—	—	—	68·4	69·3	—	—	—	—	—	—	—	—	—	—	—
	35·0	—	51·0	40·0	95·0	—	—	74·3	75·1	69·0	107·8	73·6	51·9	49·0	80·0	83·3	78·4	—	59·6	87·0
	29·6	—	47·7	37·8	96·5	34·1	31·3	76·3	76·7	76·0	100·4	69·7	50·2	51·2	70·9	74·9	79·2	91·8	—	—
	—	—	—	41·1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
	34·7	—	41·1	35·2	—	—	—	—	—	—	—	69·6	—	47·4	81·6	80·3	85·6	—	—	—

d in calculating the mean values of the characters. Dioptographic tracings of its contours have been published by Parsons.

LE ANGLO-SAXON CRANIA FROM BIDFORD-ON-AVON.

	fml	*fmb*	100 *B/L*	100 *B/F*	100 *H'/L*	100 *B/H'*	100 *G'H/GB*	100 *G'H/J*	100 *NB/NH'*	100 $O_2/O_1, R$	100 $O_2/O_1, L$	100 G_2/G_1	100 *fmb/fml*	*Oc. I*	∠*P*	∠*N*	∠*A*	∠*B*	∠θ_1	∠θ_2	
o	43·0	37·5	77·7	78·5	71·1	109·3	80·3	52·8	44·2	81·4	84·4	98·9	87·2	56·0	88·0	61·0	77·0	42·0	31·0	11·0	Considerab asymme
o	40·0	30·0	72·9	74·0	70·4	103·6	65·3	47·5	47·5	—	77·2	87·5	75·0	57·3	83·0	71·0	73·0	36·0	26·0	10·0	Left infrani long. H
5	40·0	33·0	72·8	74·1	72·8	100·0	79·5	55·8	48·0	88·9	84·9	89·8	82·5	58·6	89·0	61·5	76·5	42·0	29·5	12·5	Double epi
o	46·0	35·0	71·7	72·5	69·6	103·0	70·6	53·5	40·4	87·7	82·8	89·5	76·1	57·7	88·0	67·5	72·5	40·0	24·5	15·5	Nasal bone
	—	—	77·4	78·4	—	—	—	—	—	—	—	—	—	56·8	—	—	—	—	—	—	—
	—	—	—	—	—	—	—	—	42·0	—	—	—	—	—	—	—	—	—	—	—	—
	—	—	—	—	—	—	—	—	—	—	—	—	—	56·9	—	—	—	—	—	—	—
5	33·5	31·5	77·7	—	—	—	69·9	47·1	49·0	—	81·5	83·2	94·0	60·6	83·0	72·0	72·0	36·0	25·0	11·0	Vault abse
5	41·0	34·0	70·3	71·1	71·4	98·5	76·4	48·9	43·2	87·7	85·7	76·5	82·9	60·2	83·0	66·0	73·0	41·0	31·0	10·0	Nasal bone
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
	—	—	—	—	—	—	73·6	—	43·1	84·8	89·7	78·1	—	—	84·0	—	—	—	—	—	Metopic. N
	35·0	32·0	75·4	76·2	—	—	—	—	—	—	—	—	91·4	55·5	—	—	—	—	—	—	—
	—	—	—	—	—	—	—	—	—	—	94·6	—	—	60·9	—	—	—	—	—	—	Very promi
	—	—	—	—	—	—	71·5	50·8	45·8	89·9	91·1	73·4	—	—	84·0	—	—	—	—	—	Asymmetri
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
	—	—	—	—	—	—	—	—	—	80·6	—	—	—	—	—	—	—	—	—	—	Four small
	—	—	—	—	—	—	70·6	52·3	41·1	84·1	79·5	75·3	—	—	87·0	—	—	—	—	—	Ossified tro
5	36·0	30·0	71·1	71·6	70·0	101·5	72·7	52·2	—	—	84·0	75·2	83·3	56·6	—	66·5	73·5	40·0	—	—	Slight pren
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
	—	—	—	—	—	—	67·7	—	47·6	—	90·8	78·7	—	—	—	—	—	—	—	—	Obelionic d
	—	—	70·7	70·7	—	—	—	—	—	—	97·4	—	—	57·0	—	—	—	—	—	—	Reconstruc
	—	—	—	—	—	—	—	—	42·9	—	—	89·9	—	—	—	—	—	—	—	—	Prenasal fo
o	—	—	77·8	78·6	76·2	102·1	70·9	51·1	48·0	84·3	82·9	—	—	—	—	70·0	70·0	40·0	—	—	Nasal bone
	38·5	29·5	71·9	72·8	72·2	99·6	—	—	51·5	80·5	—	91·5	76·6	58·0	—	—	—	—	—	—	Very well-r
5	36·0	31·0	71·9	73·9	70·4	102·1	69·0	50·4	51·0	75·0	74·1	71·3	86·1	57·2	83·0	68·0	75·0	37·0	29·0	8·0	Strong supe
	—	—	—	—	—	—	—	—	—	—	—	84·1	—	—	—	—	—	—	—	—	—
	—	—	69·5	70·3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
	—	—	74·2	74·5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
4	41·6	31·2	75·8	76·5	72·2	105·0	79·1	—	47·8	—	82·9	78·0	75·0	—	—	69·0	69·0	42·0	—	—	—
	—	—	71·4	72·2	—	—	68·9	—	58·6	79·3	84·4	81·8	—	—	—	—	—	—	—	—	—
8	37·7	30·1	72·1	72·8	72·3	99·6	78·7	57·5	43·7	78·6	79·0	78·6	79·8	—	—	61·5	77·5	41·0	—	—	—
	—	—	74·6	75·2	—	—	65·5	—	54·7	76·9	72·6	—	—	—	—	—	—	—	—	—	—

these skulls have been published by Parsons.

ALE ANGLO-SAXON CRANIA FROM BIDFORD-ON-AVON.

	fml	*fmb*	100 *B/L*	100 *B/F*	100 *H'/L*	100 *B/H'*	100 *G'H/GB*	100 *G'H/J*	100 *NB/NH'*	100 $O_2/O_1, R$	100 $O_2/O_1, L$	100 G_2/G_1	100 *fmb/fml*	*Oc. I*	∠*P*	∠*N*	∠*A*	∠*B*	∠θ_1	∠θ_2	
	—	—	75·9	76·1	70·6	107·6	—	—	—	—	—	96·4	—	58·8	87·0	—	—	—	—	—	Infranasal
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
	—	—	76·4	76·8	—	—	—	—	—	84·4	—	—	—	—	83·5	—	—	—	—	—	Distorted.
	—	—	—	—	—	—	69·5	—	44·0	84·1	88·5	88·0	—	—	—	—	—	—	—	—	Third mola
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	Obelionic d
o	—	—	74·9	75·3	66·8	112·2	72·6	51·5	46·1	89·5	89·0	80·4	—	56·0	85·0	66·5	73·0	40·5	28·5	12·0	Slight obeli
o	32·5	26·5	74·0	74·0	76·9	96·3	64·4	—	54·0	83·8	86·5	83·3	81·5	62·9	80·0	70·0	77·0	33·0	30·0	3·0	Permanent
o	40·0	30·0	73·9	74·1	76·6	96·5	74·9	—	53·9	81·7	84·1	83·8	75·0	63·9	80·0	65·0	76·0	39·0	35·0	4·0	Reconstruc
	—	—	70·1	70·8	—	—	73·9	—	46·5	—	88·5	74·7	—	—	—	—	—	—	—	—	Metopic.
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	Long obeli
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
o	38·0	30·0	76·9	78·4	71·3	107·8	77·1	53·3	48·5	88·5	—	76·2	78·9	63·5	—	64·0	77·0	39·0	—	—	Large left asymme
5	—	—	—	—	68·8	—	—	—	46·0	—	89·7	87·5	—	—	—	63·0	75·5	41·5	—	—	—
	—	—	68·4	69·3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
o	—	—	74·3	75·1	69·0	107·8	73·6	51·9	49·0	80·0	83·3	78·4	—	59·6	87·0	62·0	79·0	39·0	—	—	—
5	34·1	31·3	76·3	76·7	76·0	100·4	69·7	50·2	51·2	70·9	74·9	79·2	91·8	—	—	73·0	69·0	38·0	—	—	—
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
	—	—	—	—	—	—	69·6	—	47·4	81·6	80·3	85·6	—	—	—	—	—	—	—	—	—

characters. Dioptographic tracings of its contours have been published by Parsons.

θ_2	Remarks.
1·0	Considerably damaged and reconstructed. Nasal bones asymmetrical. Malars transversely grooved. Base distinctly asymmetrical. Left occipital region projects. Right mastoid slightly larger than left. Slight metopic ridge.
0·0	Left infranasal border of infantile type. Left squamoso-frontal suture 16 mm. long. Right spheno-parietal suture 5·5 mm. long. Healed aperture of irregular outline at antero-inferior angle of left parietal. Right jugular foramen very large.
2·5	Double epipteric ossicle on right side. Sagittal crest. Marked asymmetry. Small right paroccipital process.
5·5	Nasal bones very asymmetrical. Right paramastoid process. Marked metopic boss. Three lambdoid ossicles.
—	—
—	—
—	—
1·0	Vault absent. Broad asymmetrical nasal bones.
0·0	Nasal bones asymmetrical. Well-marked prenasal fossae. Bregmatic ossicle. Large left asterionic ossicle.
—	—
—	Metopic. Nasal bones synostosed at tips.
—	—
—	Very prominent superciliary ridges uniting with supraorbital to form incipient torus.
—	Asymmetrical nasals. Complete carotico-clinoid foramina. Epipteric ossicles.
—	—
—	Four small sutural bones at lambda.
—	Ossified trochleae. Nasal bones asymmetrical. Occipital region distorted.
—	Slight prenasal fossae. Slight obelionic depression. Teeth very much worn.
—	—
—	Obelionic depression.
—	Reconstructed. Metopic. United anterior bregmatic ossicle. Right coronal ossicle. Small sutural bones at lambda.
—	Prenasal fossae.
—	Nasal bones very narrow. Right epipteric ossicle.
—	Very well-marked prenasal fossae. Marked occipital asymmetry. Slight metopic ridge.
8·0	Strong superciliary ridges. Three lambdoid ossicles.
—	—
—	—
—	—
—	—
—	—
—	—
—	—

θ_2	Remarks.
—	Infranasal grooves of anthropoid type.
—	—
—	Distorted. Two small sutural bones in left lambdoid suture.
—	Third molars not present.
—	Obelionic depression.
2·0	Slight obelionic depression. Lambdoid depressions.
3·0	Permanent canines erupting on palate.
4·0	Reconstructed. Asymmetrical nasals. Left prenasal fossa.
—	Metopic.
—	Long obelionic depression. Prenasal fossae.
—	—
—	Large left asterionic ossicle. Bregmatic ossicle, 19 mm. long almost entirely behind the coronal suture. Marked asymmetry in occipital region.
—	—
—	—
—	—
—	—
—	—
—	—
—	—

To face p. 386.

APP

Serial Letter or No.	
A	
F	12
F_1	
K	
V	12
2*a*	12
3	11
5	12
17	11
48	
50	11
58	10
60	
66	13
70	11
78	10
82	12
84	11
92	11
106	11
107	12
113	
116	
118	
127	
147	12
157	
169	12
172	
183	10
185	11

APPE

Serial Letter or No.	
E	
G	10
L	
2	
13	11
27	12
33	
46	
56 i	10
56 ii	12
63	11
75	11
89	
90	
115	11
121	
152	12
178	10
181	11

APPENDIX III. INDIVIDUAL MEASUREMENTS OF MALE ANGLO-SAXON MANDIBLES FROM BIDFORD-ON-AVON.

Serial Letter or No.	w_1	g_0g_0	c_rc_r	zz	c_yl	ml	c_yl	rb'	m_2p_1	h_1	m_2h	c_rh	rl	$\angle M$	$\angle R$	$\angle C'$	100 c_rh/ml	100 c_rc_r/ml	100 g_0g_0/c_yl	100 g_0g_0/c_rc_r	100 rb'/rl
A	—	108·0	104·5	48·0	22·0	102·0	82·0	32·0	29·5	32·5	31·0	81·0	67·5	111·0	89·0	72·5	79·4	102·5	131·7	103·3	47·4
F	123·5	92·0	105·0	45·5	20·5*	105·0	86·0	34·0	28·5	38·0	29·5	76·0*	71·0*	110·0	82·0	67·0	72·4	100·0	107·0	87·6	47·9
F_1	—	—	—	48·5	—	—	82·0	36·5*	28·0	31·0	32·5	69·0*	—	115·0	—	60·0	—	—	—	—	—
K	—	93·5	95·0	—	—	101·0	79·0	33·0*	27·5	32·5	25·5	63·0	—	115·0	—	67·0	62·4	94·1	118·4	98·4	—
Y	125·0	95·0	106·0	45·0	24·0	107·0	79·0	33·0	29·5	35·5	30·5	75·0	70·0	116·0	79·0	73·0	70·1	99·1	120·3	89·6	47·1
2*a*	125·0	99·5	108·0	48·0	22·5	107·0	82·c	36·0	27·5	29·0	28·5	70·0	70·0	116·0	74·0	66·5	65·4	100·9	121·3	92·1	51·4
3	114·0	94·0	102·0	44·5	20·5	103·0	76·0	30·5	27·5	32·5	27·0	68·5	60·0	120·0	78·0	66·0	66·5	99·0	123·7	92·2	50·8
5	123·0	96·0	107·5	48·5	21·5	104·0	78·0	32·0	29·0	31·0	27·5	72·0	73·0	115·0	72·0	79·0	69·2	103·4	123·1	89·3	43·8
17	116·5	82·0	97·5	40·5	19·5	101·0	75·0	29·0	—	27·5	—	63·5	60·0	121·0	73·0	59·0	62·9	96·5	109·3	84·1	48·3
48	—	81·0	—	43·5	—	122·0	76·0	32·0*	27·5	31·5	28·5	66·0*	71·0*	135·0	60·0	—	54·1	—	106·6	—	45·1
50	115·5	85·5	87·5	44·0	—	101·0	73·5	30·5	29·5	30·0	28·0	66·0*	68·5*	118·0	68·0	70·0	65·3	86·6	116·3	97·7	44·5
58	102·0	92·0	86·0	42·5	—	110·0	74·0	29·0*	29·5	30·0	27·5	57·0*	60·0*	128·0	61·0	66·0	51·8	78·2	124·3	107·0	48·3
60	—	—	—	43·0	—	—	78·0	37·5*	28·0	31·0	28·5*	62·0*	—	120·0	—	71·0	—	—	—	—	—
66	130·5	103·0	105·0	50·5	20·0*	100·0	82·0	32·5	28·5	33·0	29·0	74·0	66·5*	110·0	88·0	62·5	74·0	105·0	125·6	98·1	48·9
70	110·5	96·0	88·0	42·5	—	92·0	61·0	26·5	28·0	26·0	23·0	60·0	60·0	125·0	69·0	60·0	54·3	95·7	157·4	109·1	44·2
78	108·5	102·5	94·0	44·0	18·0	107·0	77·0	27·5	—	27·5	—	57·0*	57·0	129·0	—	—	53·3	87·9	133·1	109·0	48·2
82	127·0	106·5	—	51·0	—	110·0	77·0	33·0	27·5*	34·0	31·0*	65·5	60·0	124·0	77·0	68·0	59·5	—	138·3	—	55·0
84	116·0	92·0	97·0	51·0	22·5	112·0	79·0	33·0	30·5	39·5	33·0	—	62·0	125·0	—	75·0	—	86·6	116·5	94·8	53·2
92	112·5	90·0	92·0	44·5	18·5	111·5	78·0	34·5	26·5	36·5	35·5	66·5	72·5	120·0	64·0	71·0	57·8	80·0	115·4	97·8	47·6
106	118·5	101·5	91·5	43·5	18·5	103·0	77·5	32·0	29·0	29·5	30·0	56·0	64·0	113·0	55·0	68·5	54·4	88·8	131·0	110·9	50·0
107	120·0	97·0	97·0	49·5	23·5	109·0	86·0	35·0	29·0	34·0	30·5	76·0	71·5	116·0	73·5	69·0	69·7	89·0	112·8	100·0	49·0
113	—	94·5	100·5	46·0	—	111·0	83·0	34·0	29·0	33·5	27·5	64·0	—	121·0	—	68·5	57·7	90·5	113·9	94·0	—
116	—	95·5	99·0	46·5	16·5	112·0	82·0	32·5	28·5	35·5	29·5	—	67·5	121·0	—	64·5	—	88·4	116·5	96·5	48·1
118	—	—	99·5	48·5	18·0*	113·5	79·0	33·0*	26·5	32·5	30·0	80·0*	71·0*	123·0	82·0	—	70·5	87·7	—	—	46·5
127	—	103·0	—	48·0	23·5	105·0	73·0	32·0	27·5	32·5	29·5	—	66·5	127·0	—	73·5	—	—	141·1	—	48·1
147	124·0	102·5	—	49·0	20·5	106·0	82·0	36·0	28·5*	32·0	29·5	66·0*	64·0	117·0	73·0	69·0	62·3	—	125·0	—	56·3
157	—	92·5	85·0	42·5	16·0*	101·0	76·0	32·0*	26·5	34·0	23·0	55·0*	52·0*	124·0	68·0	66·5	54·5	84·2	121·7	108·8	61·5
169	127·0	103·0	103·0	46·5	23·0	107·0	75·0	33·0	31·0	38·0	31·0	64·0	63·0	123·0	69·0	73·0	59·8	96·3	137·3	100·0	52·4
172	—	—	—	45·5	—	—	—	—	26·0*	30·0	28·0*	63·5*	—	—	—	73·0	—	—	—	—	—
183	102·0	80·5	83·0	44·0	19·5	101·0	74·0	32·5	30·5	28·0	24·0	57·0	64·0	117·0	60·0	68·5	56·4	82·2	108·8	97·0	50·8
185	112·0	90·0	104·0	47·5	22·0	106·0	82·0	34·5	26·5	31·5	30·0	72·0	70·0	115·0	72·0	69·0	67·9	98·1	109·8	86·5	49·3

* These measurements were taken on the right side.

APPENDIX IV. INDIVIDUAL MEASUREMENTS OF FEMALE ANGLO-SAXON MANDIBLES FROM BIDFORD-ON-AVON.

Serial Letter or No.	w_1	g_0g_0	c_rc_r	zz	c_yl	ml	c_yl	rb'	m_2p_1	h_1	m_2h	c_rh	rl	$\angle M$	$\angle R$	$\angle C'$	100 c_rh/ml	100 c_rc_r/ml	100 g_0g_0/c_yl	100 g_0g_0/c_rc_r	100 rb'/rl
E	—	89·0	91·5	45·5	—	—	74·0	31·0	28·0	28·5	28·0	66·0*	—	116·0	—	65·0	—	—	120·3	97·3	—
G	102·0	85·0	89·5	44·5	18·0*	103·0	72·0	30·0	29·0	27·5	25·5	63·0*	58·0*	125·0	76·0	69·5	61·2	86·9	118·1	95·0	51·7
L	—	88·5	—	41·0	23·0*	107·0	77·0	31·5	—	27·0	—	64·0	57·0*	123·0	—	62·5	59·8	—	114·9	—	55·3
2	—	—	102·0	45·5	—	—	—	31·0	27·5	31·5	26·5	—	—	—	—	71·5	—	—	—	—	—
13	112·0	91·5	92·0	41·5	—	103·0	78·0	26·0	28·0	32·5	25·5	55·0*	65·5*	120·0	70·0	65·5	53·4	89·3	117·3	99·5	39·7
27	125·5	100·5	100·0	51·0	20·0	104·0	81·0	36·0	28·5	30·0	30·5	66·0	59·0	118·0	73·5	73·0	63·5	96·2	124·1	100·5	61·0
33	—	86·0	104·0	46·0	—	94·0	73·0	32·5	28·0	30·0	27·0	65·0	60·0	116·0	78·0	77·0	69·1	110·6	117·8	82·7	54·2
46	—	98·0	95·5	44·5	20·0	105·0	79·0	34·0	26·0*	29·0	26·0	60·0	57·5	130·0	66·0	—	57·1	91·0	124·1	102·6	59·1
56 i	102·0	78·5	90·0	44·5	16·0	97·0	73·0	30·0	27·5	27·5	24·0	57·0	61·0	115·0	63·0	74·0	58·8	82·5	107·5	87·2	49·2
56 ii	125·0	91·5	94·0	49·0	21·0	107·0	81·0	32·5	28·5	30·0	30·5	62·0*	67·0	115·0	61·0	73·5	57·9	87·9	113·0	97·3	48·5
63	116·5	—	89·0	46·0	20·5	112·0	82·0	37·0*	26·5	30·5	29·5	63·0*	63·0	122·0	64·0	66·5	56·3	79·5	—	—	58·7
73	112·0	90·0	—	45·5	20·0	91·0	76·0	31·0	25·5*	26·0	26·0	66·0*	66·0	105·0	80·0	71·0	72·5	—	118·4	—	47·0
89	—	85·5	90·0	46·0	—	90·0	71·0	32·5	29·5	26·0	26·0	62·0	—	120·0	—	60·0	68·9	100·0	120·4	95·0	—
90	—	—	—	45·5	—	102·0	73·0	30·0	27·0	29·0	26·0	58·0	—	120·0	—	67·0	56·9	—	—	—	—
115	112·0	98·0	91·5	49·0	21·0	107·0	77·0	29·0	29·0	30·5	27·0	68·0	60·0	125·0	68·0	73·0	63·6	85·5	127·3	107·1	48·3
121	—	102·5	103·0	52·5	20·5*	106·0	80·0	32·5*	29·5	31·5	26·5	67·0*	67·0*	117·0	74·0	73·5	63·2	97·2	128·1	99·5	48·5
152	121·0	91·0	90·0	49·0	24·0	109·0	84·0	37·0	28·5	30·5	27·5	65·0	62·0	120·0	74·0	65·0	59·6	82·6	108·3	101·1	58·1
178	108·0	85·5	87·5	45·0	19·5	104·0	79·0	33·0	27·0	30·0	25·0	49·0	51·0	123·0	61·0	78·5	47·1	84·1	108·2	97·7	64·7
181	119·0	92·0	95·0	44·0	21·5*	106·5	72·0	25·0*?	27·5	30·5	25·0	60·0	68·0*	128·0	65·0	—	56·3	89·2	127·8	96·8	36·8?

* These measurements were taken on the right side.

PART II. THE BURWELL SKULLS, INCLUDING A COMPARISON WITH THOSE OF CERTAIN OTHER ANGLO-SAXON SERIES.

By Doris Layard and Matthew Young.

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APPENDIX TO PART II.

V. Individual Measurements of the Male Anglo-Saxon Crania from Burwell.
VI. Individual Measurements of the Female Anglo-Saxon Crania from Burwell.
VII. Individual Measurements of the Male Anglo-Saxon Mandibles from Burwell.
VIII. Individual Measurements of the Female Anglo-Saxon Mandibles from Burwell.

1. *Introduction.* Burwell lies to the north-east of Cambridge, on the borders of Cambridgeshire and Suffolk. The presence of a burial-place in this situation was first discovered in 1884, when fourteen skeletons with funerary objects were unearthed in the process of working the Victoria Lime Pits which are cut in the side of a low chalk hill that rises here out of the Fen. The land adjoining the pits is let out to allotment holders and belongs to Dr Charles Lucas, who, in 1925, requested the Cambridge Antiquarian Society to undertake excavations. These were carried out in five successive seasons under the direction of Mr T. C. Lethbridge, who published interim reports of his results in the *Cambridge Antiquarian Society's Communications*, Vols. XXVII, XXVIII, XIX and XXX, 1926—1929. A detailed account of them appears in *Recent Excavations in Anglo-Saxon Cemeteries in Cambridgeshire and Suffolk**.

* T. C. Lethbridge. *Cambridge Antiquarian Society Quarto Publications*, New Series, No. III. Cambridge, 1931.

During the excavations about 125 graves were exposed. The skeletal material was deposited in the Anatomy Museum at the University of Cambridge, and the grave-goods are preserved in the University Museum of Archaeology and Ethnology.

The graves were scattered irregularly about the site, and in no case encroached on one another; Mr Lethbridge suggests that each may have been covered formerly by a low mound or barrow, which would have served to mark its position. The graves were shallow and cut in the chalk, their depth being determined by the thickness of the soil above the chalk. In the majority, the bodies were orientated with heads towards the west. The grave-goods were poor both in quantity and quality, and are remarkable for the fact that the ornaments are all of Kentish workmanship and belong to types known to be prevalent in Kent at the end of the Pagan period.

After reviewing in detail the material from Burwell and comparing it with material from other Anglo-Saxon cemeteries in the neighbourhood, Mr Lethbridge has come to the conclusion that the Burwell cemetery is that of a Christian Anglo-Saxon community of the seventh century.

We are indebted to Professor H. A. Harris and Dr W. L. H. Duckworth for granting permission and providing facilities for the measurement of the skulls.

2. *Cranial Material available for Investigation and Measurement.* Though the skeletons that were excavated were numbered serially from 1 to 140, the number of adult crania that were sufficiently complete after reconstruction to permit of some at least of their principal measurements being taken was only 67. Of this number 45 were considered to be male and 22 female. The measurements of the juvenile skulls, some of which were considerably distorted, were all excluded from the averages. The principal measurements of the individual crania are tabulated in detail in the Appendices V and VI, and the mean values of these characters for male and female groups of crania are given in Tables I and II. For the male series, where the number of crania on which the mean values of characters are based is usually in excess of 30 and frequently in the neighbourhood of 40, the standard deviations and coefficients of variation of the different characters have been calculated; to these mean values and standard deviations their standard errors are appended. As the means of the characters in the female series are, with a few exceptions, based on less than 20 observations, the standard deviations have not been deemed sufficiently representative for insertion in the table. For comparison with the mean values of the cranial characters of these Burwell Anglo-Saxons and of the Bidford-on-Avon Anglo-Saxon skulls measured by J. C. Brash and described in Part I of this paper, the corresponding means of the Anglo-Saxon crania from the London Museums which were measured and described by G. M. Morant* have been tabulated.

The Burwell male series of crania may be considered fairly homogeneous, since the standard deviations of a selection of the principal characters are not larger than

* *Biometrika*, Vol. XVIII. (1926), p. 56.

TABLE I.

Comparative Table of Means for Male Crania.

Characters	Nos.	Burwell			London Museums	Bidford-on-Avon
		Means	σ	v	Means	Means
C	40	[1539·12 ± 10·24]	—	—	[1543·3] (31)	[1595·7] (11)
F	44	187·19 ± 0·78	5·18 ± 0·55	2·77	188·9 (94)	190·3 (19)
L	45	189·62 ± 0·84	5·60 ± 0·59	2·95	190·6 (58)	192·8 (19)
B	45	141·68 ± 0·82	5·50 ± 0·58	3·88	141·7 (103)	141·7 (19)
B'	44	95·22 ± 0·51	3·40 ± 0·36	3·57	97·3 (59)	96·6 (24)
H'	40	136·29 ± 0·91	5·75 ± 0·64	4·22	136·0 (31)	139·7 (11)
OH	37	114·09 ± 0·81	4·95 ± 0·58	4·34	114·9 (17)	115·6 (9)
LB	40	102·38 ± 0·83	5·26 ± 0·59	5·14	104·1 (31)	106·2 (12)
Q	37	312·86 ± 1·86	11·32 ± 1·32	3·62	314·4 (15)	—
Q'	36	317·06 ± 1·90	11·37 ± 1·34	3·59	316·1 (23)	307·3 (15)
S	42	381·64 ± 1·58	10·25 ± 1·12	2·69	379·8 (64)	382·4 (12)
S_1	42	128·38 ± 0·94	6·06 ± 0·66	4·72	129·3 (83)	128·1 (20)
S_2	42	130·86 ± 1·13	7·31 ± 0·80	5·59	129·0 (87)	131·0 (18)
S_3	42	122·55 ± 1·15	7·44 ± 0·81	6·07	121·7 (71)	121·7 (14)
S_1'	42	112·69 ± 0·73	4·76 ± 0·52	4·22	—	112·7 (24)
S_2'	42	117·36 ± 0·89	5·74 ± 0·63	4·89	—	117·1 (19)
S_3'	42	99·79 ± 0·85	5·52 ± 0·60	5·53	98·5 (37)	99·8 (11)
U	42	527·76 ± 2·01	13·05 ± 1·42	2·47	532·0 (73)	537·1 (11)
GH	34	115·84 ± 1·14	6·65 ± 0·81	5·74	—	118·0 (12)
$G'H$	31	69·02 ± 0·90	5·04 ± 0·64	7·30	71·7 (22)	69·5 (21)
GB	31	94·71 ± 1·02	5·68 ± 0·72	6·00	95·0 (19)	95·3 (17)
J	24	134·02 ± 1·31	6·40 ± 0·92	4·78	133·3 (34)	136·0 (12)
NH'	34	50·15 ± 0·60	3·51 ± 0·43	7·00	52·7 (28)	51·7 (19)
NH, R	34	50·12 ± 0·62	3·61 ± 0·44	7·20	52·2 (22)	—
NH, L	34	50·04 ± 0·62	3·63 ± 0·44	7·25	52·3 (22)	—
NB	34	24·12 ± 0·31	1·80 ± 0·22	7·46	24·5 (28)	24·1 (19)
O_1', R	12	39·67 ± 0·61	2·10 ± 0·43	5·29	40·3 (16)	—
O_1, R	30	42·02 ± 0·34	1·86 ± 0·24	4·43	42·9 (19)	40·4 (14)
O_1, L	33	41·85 ± 0·30	1·71 ± 0·21	4·09	42·2 (16)	40·5 (19)
O_2, R	30	32·78 ± 0·42	2·30 ± 0·30	7·02	33·6 (29)	33·4 (14)
O_2, L	33	32·77 ± 0·35	2·02 ± 0·25	6·16	33·5 (27)	34·1 (19)
G_1	11	48·50 ± 1·05	3·49 ± 0·74	7·20	50·1 (20)	47·4 (18)
G_1'	14	44·57 ± 0·87	3·26 ± 0·62	7·31	—	—
G_2	29	40·41 ± 0·48	2·59 ± 0·34	6·41	41·3 (27)	38·9 (18)
GL	33	94·11 ± 1·01	5·79 ± 0·71	6·15	96·0 (22)	101·9 (11)
fml	37	37·32 ± 0·60	3·63 ± 0·42	9·73	37·5 (20)	39·0 (12)
fmb	37	31·11 ± 0·39	2·38 ± 0·28	7·65	31·1 (18)	32·1 (12)
100 B/L	45	74·76 ± 0·47	3·16 ± 0·33	—	74·7 (52)	73·5 (19)
100 B/F	44	75·75 ± 0·45	3·00 ± 0·32	—	74·9 (112)	74·1 (18)
100 H'/L	40	71·87 ± 0·51	3·22 ± 0·36	—	71·2 (25)	71·7 (11)
100 B/H'	40	104·43 ± 0·82	5·19 ± 0·58	—	104·9 (61)	102·2 (11)
100 $G'H/GB$	30	72·46 ± 1·07	5·88 ± 0·76	—	75·3 (43)	72·4 (17)
100 $G'H/J$	21	51·85 ± 0·88	4·02 ± 0·62	—	—	—
100 NB/NH, R	34	48·32 ± 0·74	4·31 ± 0·52	—	48·1 (19)	—
100 NB/NH, L	34	48·39 ± 0·75	4·39 ± 0·53	—	47·9 (19)	—
100 NB/NH'	34	48·28 ± 0·54	4·43 ± 0·76	—	47·5 (58)	46·9 (19)
100 O_2/O_1, R	30	78·11 ± 1·03	5·64 ± 0·73	—	77·9 (19)	82·8 (14)
100 O_2/O_1, L	33	78·34 ± 0·71	4·06 ± 0·50	—	79·4 (16)	84·2 (19)
100 fmb/fml	37	83·84 ± 1·29	7·85 ± 0·91	—	82·3 (18)	82·5 (12)
100 G_2/G_1	11	82·95 ± 1·71	5·68 ± 1·21	—	81·5 (18)	82·3 (18)
$\angle P$	34	87°·07 ± 0·56	3°·24 ± 0·39	—	88°·1 (16)	85°·2 (10)
$\angle N$	32	63°·30 ± 0·61	3°·47 ± 0·43	—	62°·1 (16)	66°·7 (11)
$\angle A$	32	75°·80 ± 0·60	3°·37 ± 0·42	—	75°·5 (16)	73°·6 (11)
$\angle B$	32	40°·91 ± 0·49	2°·77 ± 0·35	—	42°·4 (16)	39°·7 (11)
$\angle \theta_1$	32	29°·88 ± 0·55	3°·09 ± 0·39	—	30°·6 (13)	28°·0 (7)
$\angle \theta_2$	32	11°·09 ± 0·64	3°·62 ± 0·45	—	11°·9 (13)	11°·1 (7)
Oc. I.	42	58·94 ± 0·44	2·83 ± 0·31	—	58·2 (35)	57·8 (14)

All length measurements are in millimetres and capacities in cubic centimetres.

TABLE II.

Comparative Table of Means for Female Crania.

Characters	Burwell	London Museums	Bidford-on-Avon
	Means	Means	Means
C	[1399·1] (13)	[1370·0] (28)	[1411·0] (10)
F	182·0 (21)	180·9 (62)	182·8 (13)
L	182·9 (22)	182·0 (55)	184·0 (14)
B	138·6 (20)	135·6 (67)	136·4 (10)
B'	95·9 (18)	94·3 (58)	95·5 (15)
H'	129·3 (13)	129·6 (28)	131·9 (7)
OH	111·6 (14)	110·5 (19)	111·2 (5)
LB	95·0 (13)	97·4 (26)	99·7 (7)
Q	305·1 (14)	—	—
Q'	309·7 (14)	301·7 (18)	303·3 (6)
S	366·5 (19)	366·5 (38)	367·1 (5)
S_1	124·0 (20)	124·4 (63)	123·0 (12)
S_2	126·0 (19)	123·2 (57)	127·3 (10)
S_3	116·6 (18)	115·2 (46)	119·6 (5)
S_1'	109·3 (20)	—	108·3 (17)
S_2'	114·1 (20)	—	114·9 (15)
S_3'	96·1 (18)	97·1 (33)	100·0 (6)
U	511·8 (14)	510·9 (54)	518·0 (6)
GH	109·8 (7)	—	110·3 (9)
G'H	65·3 (12)	65·9 (30)	65·4 (9)
GB	87·2 (13)	90·2 (26)	91·0 (8)
J	126·4 (5)	125·6 (26)	125·2 (3)
NH'	47·3 (12)	47·1 (32)	49·0 (9)
NH, *R*	47·3 (12)	48·4 (29)	—
NH, *L*	47·3 (12)	48·0 (27)	—
NB	22·8 (11)	24·2 (24)	23·8 (9)
O_1, *R*	41·0 (8)	41·6 (23)	40·0 (8)
O_1, *L*	39·8 (8)	—	39·8 (8)
O_2, *R*	33·3 (8)	32·8 (27)	33·5 (8)
O_2, *L*	33·1 (8)	32·9 (26)	33·9 (8)
G_1	46·2 (3)	51·2 (13)	45·1 (10)
G_1'	43·3 (3)	47·0 (13)	—
G_2	37·6 (10)	39·9 (23)	38·2 (12)
GL	89·8 (8)	94·0 (19)	96·2 (6)
fml	36·2 (13)	35·5 (19)	36·2 (4)
fmb	29·2 (11)	28·9 (19)	29·5 (4)
100 *B*/*L*	75·8 (20)	74·4 (51)	73·8 (10)
100 *B*/*F*	76·3 (19)	75·2 (77)	74·2 (10)
100 *H'*/*L*	70·6 (12)	71·5 (27)	72·4 (7)
100 *B*/*H'*	108·4 (12)	105·7 (42)	103·5 (6)
100 *G'H*/*GB*	74·8 (12)	72·1 (41)	71·5 (8)
100 *G'H*/*J*	51·4 (5)	—	—
100 *NB*/*NH*, *R*	48·3 (11)	51·0 (20)	—
100 *NB*/*NH*, *L*	48·4 (11)	51·1 (20)	—
100 *NB*/*NH'*	48·6 (11)	50·2 (38)	48·6 (9)
100 O_2/O_1, *R*	81·3 (8)	79·6 (23)	83·1 (8)
100 O_2/O_1, *L*	82·2 (7)	79·4 (24)	85·2 (8)
100 *fmb*/*fml*	82·5 (11)	81·7 (18)	81·8 (4)
100 G_2/G_1	83·4 (3)	76·1 (12)	83·5 (10)
∠ *P*	86°·9 (13)	83°·9 (20)	83°·1 (5)
∠ *N*	65°·9 (8)	67°·2 (15)	66°·9 (6)
∠ *A*	73°·1 (8)	72°·6 (15)	74°·6 (6)
∠ *B*	41°·1 (8)	40°·1 (15)	38°·5 (6)
∠ θ_1	28°·8 (8)	29°·0 (11)	31°·2 (3)
∠ θ_2	12°·3 (8)	10°·2 (11)	6°·3 (3)
Oc. I.	59·5 (18)	59·7 (32)	61·0 (5)

All length measurements are in millimetres and capacities in cubic centimetres.

those shown by the corresponding characters in the Anglo-Saxon male skulls from the London Museums or those of the seventeenth-century Londoners which were excavated from a single pit in Whitechapel. A comparison of the values of the standard deviations in the three series for the characters selected is shown below:

Characters	Anglo-Saxons from Burwell	Anglo-Saxons from London Museums	17th-century* Londoners (Whitechapel)
L	5·60	—	6·27
F	5·18	5·49	—
B	5·50	5·79	5·28
H'	5·75	5·81	5·56
U	13·05	11·79	15·02
100 B/L	3·16	—	3·26
100 B/F	3·00	3·12	—

The Burwell female series is not only smaller but rather less homogeneous than the male. A casual inspection of the group showed that three of the skulls differed from the majority in the respect that they were definitely brachycephalic in type.

3. *Mean Values and Variabilities of the principal Cranial Characters and a Comparison with the corresponding Characters in the Anglo-Saxon Crania from Bidford-on-Avon and from the London Museums.* The mean values of the principal calvarial and facial characters in the Burwell collection of male and female skulls may be compared in detail in Tables I and II with the corresponding values in the two other groups of Anglo-Saxon skulls, but the characters in which the three cranial series diverge most notably from one another can be more readily seen from a study of Table III, in which the values of

$$\alpha = \frac{n_s n_{s'}}{n_s + n_{s'}} \left(\frac{M_s - M_{s'}}{\sigma_s} \right)^2$$

for the selected series of 31 characters, 19 absolute measurements and 12 indices and angles, which are subsequently used for the calculation of the coefficients of racial likeness, are given for each pair of male and female groups of skulls.

The general form of the skulls excavated at Burwell is illustrated by photographs in four normae of a typical male specimen (Skull No. 56). Photographs of its mandible from two aspects are also given.

4. *Values of α for a selected Series of Characters showing the Features in which Differences in the three Cranial Series are most pronounced*†. Comparing in the first place the Burwell male series with that from the London Museums, it will be seen that in only a few of the 31 characters does the value of α exceed 2·7. These divergent characters are $G'H$, B', NH, R, O_1R, LB and 100 $G'H/GB$. In the

* *Biometrika*, Vol. III. (1904), pp. 191—244.

† The standard deviations used for the calculation of the α's were those of the long Egyptian series of the XXVI—XXX dynasties. *Biometrika*, Vol. XVI. (1923), p. 328.

Norma facialis.

Norma lateralis (Right Profile).

Typical Anglo-Saxon Male Skull from Burwell (No. 56).

Norma occipitalis.

Norma verticalis.

Typical Anglo-Saxon Male Skull from Burwell (No. 56).

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Typical Male Mandible. Anglo-Saxon Skull from Burwell (No. 56).

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TABLE III.

Values of $\alpha = \frac{n_s n_{s'}}{n_s + n_{s'}} \left(\frac{M_s - M_{s'}}{\sigma_s}\right)^2$ *between the Several Cranial Series.*

Characters	Males			Females		
	Burwell and London Museums	Bidford-on-Avon and London Museums	Burwell and Bidford-on-Avon	Burwell and London Museums	Bidford-on-Avon and London Museums	Burwell and Bidford-on-Avon
100 B/L	0·03	2·79	3·14	4·30	0·46	6·25
100 H'/L	0·87	0·22	0·04	0·83	0·55	1·76
100 B/H'	0·33	3·67	2·26	4·11	1·53	5·80
100 $G'H/GB$	5·63	4·16	0·00	3·37	0·12	2·60
100 NB/NH, R	0·03	—	—	3·64	—	—
100 NB/NH'	—	0·35	1·64	—	1·31	0·00
100 O_2/O_1, R	0·02	7·59	8·27	0·81	3·45	0·62
100 fmb/fml	0·81	0·01	0·46	0·14	0·00	0·05
100 G_2/G_1	0·33	0·12	0·07	2·95	6·90	0·00
Oc. I.	0·86	0·15	1·17	0·05	0·71	0·86
$\angle P$	1°·04	4°·93	2°·66	8°·15	0°·29	5°·99
$\angle N$	1°·40	12°·59	8°·64	0°·83	0°·04	0°·32
$\angle A$	0°·08	1°·97	3°·31	0°·11	1°·47	0°·66
C	0·02	1·86	2·15	0·89	1·11	0·08
L	0·77	2·12	4·18	0·57	2·00	0·46
B	0·00	0·00	0·00	6·79	0·27	1·58
B'	6·78	0·51	1·86	2·45	1·19	0·09
OH	—	—	—	—	—	—
H'	0·06	4·39	3·94	0·04	1·55	1·61
LB	3·20	2·42	8·46	3·31	2·35	8·11
Q'	0·14	7·19	10·40	6·43	0·15	2·20
S	0·53	0·44	0·04	0·00	0·01	0·01
U	2·48	1·31	3·98	0·07	1·97	1·17
$G'H$	5·44	3·02	0·18	0·22	0·12	0·00
J	0·33	3·10	1·53	0·14	0·02	0·14
NH, R	6·91	—	—	1·52	—	—
NH'	—	1·33	3·22	—	3·75	2·20
NB	0·78	0·58	0·00	5·50	0·39	1·84
O_1, R	3·38	18·06	8·76	0·88	6·24	1·64
O_2, R	2·59	0·10	0·94	0·46	0·89	0·05
G_1	1·64	6·23	0·75	6·96	24·00	0·32
G_2	1·64	8·99	3·61	5·76	3·56	0·31
fmb	0·00	1·56	1·96	0·16	0·30	0·07
fml	0·09	2·77	4·29	0·79	0·34	0·00

Burwell series the mean measurements of all these characters are less than in the series from the London Museums. On comparing the Bidford-on-Avon males with the London Museums males the number of cranial characters in which striking divergence is shown is distinctly greater than in the previous case. The characters in which the differences in these groups are most emphasised are O_1, R, G_1, G_2, Q', H', 100 $O_2/O_1, R$, 100 $G'H/GB$, 100 B/H', and the angles N and P. O_1, R, G_1, G_2, Q' and 100 $G'H/GB$, 100 B/H' and the profile angle ($\angle P$) are all smaller in the

Bidford-on-Avon than in Morant's series. The characters H', $100\ O_2/O_1, R$ and $\angle N$, on the other hand, are greater in the Bidford series. Although the length and width of the palate G_1 and G_2 are definitely less in the Bidford than in Morant's series, the relative proportion of breadth to length, $100\ G_2/G_1$, does not differ so sensibly in the two groups. While the mean heights of the right and left orbits, O_2, R and O_2, L, show closely corresponding values in the two series, the mean measurements of orbital width, O_1, R and O_1, L, are definitely smaller in the Bidford. The mean value for O_1, R in the Bidford skull, indeed, almost coincides with that tabulated for O_1', R, the breadth measured from the dacryon, in Morant's series. As a consequence, the orbital index O_2/O_1, R in the Bidford group is on the average of much the same order as that tabulated for $100\ O_2/O_1$, R in Morant's series. In measuring orbital width, J. C. Brash used the curvature method to determine the inner terminal of the diameter and not the dacryon, and the accuracy of the measurements originally taken have been verified in certain cases from the diopto-graphic tracings of the facial normae that were prepared. The notable difference in orbital width in the two groups is thus, so far as we are aware, not due to any difference in technique but would seem to indicate a real divergence in orbital shape. The nasal angle N is on the average nearly 5° larger in the Bidford than in the Museums series. The depth of the face, GL, is in the Bidford male almost 6 mm. greater on the average than in the Museums group, and the divergence in this measurement appears to be largely responsible for the angular difference. The mean breadth of the face, GB, is much the same in the Bidford and Morant's series, but the upper face height, $G'H$, is on the average 2 mm. less in the Bidford. As a consequence, the upper facial index, $100\ G'H/GB$, is definitely less in the Bidford than in the Museums series.

The values of α between the selected series of characters in the Burwell and Bidford-on-Avon male groups show clearly that the characters in which the Burwell crania differ from the Bidford are largely the same as those in which the crania from the London Museums differ from the latter group. The principal divergences occur in the characters O_1, R, G_2, Q', LB, H', L, $100\ O_2/O_1, R$ and $\angle N$. The mean orbital width, O_1, R, and orbital index, $100\ O_2/O_1$, R, in the Burwell group are in fairly close agreement with the corresponding values in Morant's series, and show the same characteristic difference from the Bidford group. Q' is on the average about 10 mm. greater in the Burwell than in the Bidford series and the basi-nasal length, LB, is nearly 4 mm. less. The depth of the face, GL, in the Bidford series is on the average almost 8 mm. greater than in the Burwell series, and this divergence, similar to that shown from Morant's mean value, is chiefly responsible for the large size of the $\angle N$ in the Bidford as compared with the Burwell cranium. Both the glabella-occipital length (L) and basi-bregmatic height (H') are on the average rather greater in the Bidford than in the Burwell group.

As the mean values of the characters which are tabulated for the female crania in the Burwell and Bidford-on-Avon series are usually based on relatively small numbers of observations, the values of α between the selected characters in the

female groups may be considered in less detail. Comparing the Burwell series with Morant's series, the characters in which the values of α seem to indicate the most striking divergences are B, Q', NB, G_1, G_2, $100\,G_2/G_1$, $100\,B/L$, $100\,B/H'$ and the $\angle P$, though in a few other characters α slightly exceeds 2·7. In the Burwell group the maximum parietal breadth, B, is on the average 3 mm. greater than in the Museums group. This difference in breadth is largely responsible for the divergence shown in the mean indices, $100\,B/L$ and $100\,B/H'$, in both of which α exceeds the value 4. On excluding the three definitely brachycephalic skulls, to which reference has been made, the mean cephalic index in the Burwell series is identical with that in Morant's series. Q' in the Burwell group is greater than in the Museums group, but G_1, G_2 and NB are smaller. The profile angle ($\angle P$) is on the average 3° greater in the Burwell female, though in the two male groups the mean values agree fairly closely. In the Museums series of Anglo-Saxon crania the mean value of this angle in the male is about 4° in excess of that for the female. In the Burwell series the mean value of $\angle P$ in the male just exceeds that in the female. In the long Egyptian series E*, the mean profile angle ($\angle P$) in the male is 85·81 ± ·08 and in the female 84·93 ± ·09, which would suggest that when the number of observations is large the profile angle in the male does not differ greatly from that in the female, probably not more than 1°. The large sexual difference in the Anglo-Saxons from the London Museums may probably be due to the small numbers on which the averages are based and yet, as will be seen later, a similar sexual difference, though of lesser degree, is found in the Bidford-on-Avon group. The difference may really be indicative of relative prognathism in the female Anglo-Saxon, though it is not evident in the Burwell type.

A scrutiny of the values of α for the selected characters in the Museums and Bidford-on-Avon females shows that the only features in which differences that may be considered significant occur are G_1, G_2, O_1, R, NH', $100\,O_2/O_1$, R and $100\,G_2/G_1$. The mean palatal length and breadth, G_1 and G_2, in the Bidford group are apparently smaller than the corresponding values in Morant's series, and as the length is relatively more in defect than the breadth, the palatal index, $100\,G_2/G_1$, is definitely greater in the Bidford series. The orbital width, O_1, R, is less in the Bidford than in Morant's group and the orbital index apparently greater. The mean value of NH' in the Bidford group, on the other hand, is greater than in the Anglo-Saxons from the London Museums.

Comparing the Burwell and Bidford-on-Avon females, the only values of α which seem to indicate significant differences in the corresponding characters are those for LB, $100\,B/L$, $100\,B/H'$ and $\angle P$. The basi-nasal length, LB, in the Bidford female is on the average very large and exceeds the mean value for the corresponding character in the Burwell group by nearly 5 mm. A like difference was found in the male crania. The mean profile angle ($\angle P$) in the Bidford female is based on only five observations. The individual angles in this

* *Biometrika*, Vol. XVI. (1924), pp. 328—363.

group were measured on the dioptographic tracings of the normae laterales and not directly by the stationary goniometer, but this is not likely to have influenced very materially the estimation of the size of the angle, as both methods when tested on two or three male skulls gave almost identical results. The mean value of the $\angle P$, as has been mentioned, is in close agreement with that for the Anglo-Saxons from the London Museums which, though probably a smaller value than would be found if it were based on a large number of observations, may indicate a relative tendency to prognathism in the Anglo-Saxon female. In the Burwell group the mean maximum breadth is greater, and the mean maximum length and mean basi-bregmatic height less than in the Bidford. As a consequence the indices $100\ B/L$ and $100\ B/H'$ in the Burwell group both exceed the corresponding indices in the Bidford group to a greater degree than might be expected to occur fortuitously.

5. *Coefficients of Racial Likeness for the Cranial Series.* Having enumerated and discussed the characters for which the mean measurements in the groups of skulls seem to show differences of such a degree that they may probably be considered significant, we now proceed to consider the coefficients of racial likeness computed between each pair of groups of skulls for the 31 selected characters. The coefficients, crude and reduced, are shown for males in Table IV and for females in Table V. The reduced coefficients are relatively comparable as they have been brought to a common standard by making allowance for the varying number of skulls on which the crude coefficients are based.

The crude coefficients for the male groups seem to indicate quite clearly that in shape and in general characters the Burwell crania are almost identical with the Anglo-Saxon crania from the London Museums. Both these groups show a slight divergence, almost equal in degree, from the Bidford-on-Avon crania. As the mean number of skulls on which the Burwell and Museums coefficients are based is much the same, the adjustment for varying numbers of skulls does not change the relationship of the Bidford-on-Avon group to these two groups. The reduced coefficients of racial likeness between the Bidford-on-Avon male group and the Burwell and Morant's series respectively are, however, about six times the size of the coefficient between these two series.

The crude coefficients of racial likeness for the female groups, Table V, suggest that in regard to shape-characters (indices and angles) the Burwell series diverges only slightly and to an equivalent degree from the two series from Bidford-on-Avon and the London Museums. In its characters, generally, the Burwell series seems to resemble the Bidford-on-Avon series rather more closely than that from the London Museums. The London Museums cranium is almost identical in shape with the Bidford-on-Avon cranium, and diverges only slightly from it when comparison is made between all the 31 characters under review.

Though a comparison of the reduced and relatively comparable coefficients which are given in Table V suggests that in their general form the female Anglo-Saxons from the London Museums may resemble the Bidford-on-Avon crania

TABLE IV.

Coefficients of Racial Likeness between the Male Cranial Series.

	Burwell (34·8)		London Museums (35·3)		Bidford-on-Avon (14·8)	
	All Characters	Indices and Angles	All Characters	Indices and Angles	All Characters	Indices and Angles
Crude Coefficients						
Burwell (34·8) ...	—	—	0·56 ± ·17	−0·05 ± ·28	1·96 ± ·17	1·64 ± ·28
London Museums (35·3)	0·56 ± ·17	−0·05 ± ·28	—	—	2·27 ± ·17	2·21 ± ·28
Bidford-on-Avon (14·8)	1·96 ± ·17	1·64 ± ·28	2·27 ± ·17	2·21 ± ·28	—	—
Reduced Coefficients						
Burwell (34·8) ...	—	—	1·63 ± ·49	−0·16 ± ·91	9·49 ± ·82	8·32 ± 1·42
London Museums (35·3)	1·63 ± ·49	−0·16 ± ·91	—	—	10·88 ± ·82	11·74 ± 1·45
Bidford-on-Avon (14·8)	9·49 ± ·82	8·32 ± 1·42	10·88 ± ·82	11·74 ± 1·45	—	—

TABLE V.

Coefficients of Racial Likeness between the Female Cranial Series.

	Burwell (12·1)		London Museums (30·5)		Bidford-on-Avon (7·6)	
	All Characters	Indices and Angles	All Characters	Indices and Angles	All Characters	Indices and Angles
Crude Coefficients						
Burwell (12·1) ...	—	—	1·33 ± ·17	1·44 ± ·28	0·51 ± ·17	1·08 ± ·28
London Museums (30·5)	1·33 ± ·17	1·44 ± ·28	—	—	1·16 ± ·17	0·40 ± ·28
Bidford-on-Avon (7·6)	0·51 ± ·17	1·08 ± ·28	1·16 ± ·17	0·40 ± ·28	—	—
Reduced Coefficients						
Burwell (12·1) ...	—	—	7·73 ± ·99	9·11 ± 1·77	5·46 ± 1·82	12·49 ± 3·24
London Museums (30·5)	7·73 ± ·99	9·11 ± 1·77	—	—	9·54 ± 1·40	3·58 ± 2·50
Bidford-on-Avon (7·6)	5·46 ± 1·82	12·49 ± 3·24	9·54 ± 1·40	3·58 ± 2·50	—	—

more closely than they do the Burwell or than the Burwell resemble the Bidford-on-Avon group, it is obvious from the size of the probable errors of the coefficients that little emphasis can be laid on such differences as are observed. The coefficient for all characters between the Burwell and Bidford-on-Avon groups is rather less than that between the Burwell and the London Museums groups, and the latter is

less than the coefficient between the Bidford-on-Avon and the Museums groups, but it is again apparent from the size of the probable errors of the coefficients that no real differences in degree of resemblance can be postulated on such data as are available.

6. *Mandibles: Material available for Measurement.* The mandibles belonging to the Burwell skulls were also measured in accordance with the scheme of measurement now followed at the Biometric Laboratory, University College, London. 62 specimens were found to be sufficiently complete, though reconstruction was necessary in many cases, to provide measurements of the majority of the characters that are usually brought under review. Of these 42 were of the male and 20 of the female sex. The measurements of the individual mandibles are tabulated in Appendices VII and VIII.

7. *Mean Values and Variabilities of the principal Mandibular Characters and a Comparison with the corresponding Characters in the Anglo-Saxon Mandibles from Bidford-on-Avon and the London Museums, as well as the Mandibles from Dunstable.* The mean values of the characters measured and the indices computed from these for the Burwell and Bidford-on-Avon series are shown for males in Table VI and for females in Table VII. For comparison with the male series of mean values, the

TABLE VI.

Comparative Table of Means for Mandibles of Male Skulls.

Character	No.	Burwell			London Museums	Bidford-on-Avon	Dunstable
		Mean	σ	v	Mean	Mean	Mean
w_1	35	122·44 ± 1·07	6·32 ± 0·76	5·16	123·7 (25)	117·7 (20)	121·0 (29)
$g_o g_o$	41	98·82 ± 1·02	6·56 ± 0·72	6·64	100·4 (33)	95·1 (27)	98·4 (40)
$c_r c_r$	37	98·81 ± 0·79	4·83 ± 0·56	4·89	100·3 (27)	97·4 (24)	99·3 (27)
zz	42	45·75 ± 0·33	2·17 ± 0·24	4·74	45·3 (57)	46·1 (30)	45·4 (42)
$c_p l$	39	21·19 ± 0·34	2·13 ± 0·24	10·05	21·7 (38)	20·5 (21)	21·3 (35)
ml	42	105·55 ± 0·93	6·01 ± 0·66	5·69	107·2 (31)	106·1 (28)	106·8 (34)
$c_p l$	42	78·12 ± 0·80	5·21 ± 0·57	6·67	77·7 (42)	78·1 (30)	76·2 (38)
rb'	42	32·07 ± 0·49	3·18 ± 0·35	9·92	33·2 (61)	32·6 (30)	32·8 (42)
$m_2 p_1$	39	28·49 ± 0·20	1·24 ± 0·14	4·35	28·1 (59)	28·3 (29)	28·0 (39)
h_1	40	31·65 ± 0·38	2·41 ± 0·27	7·61	33·1 (40)	32·2 (31)	32·5 (39)
$m_2 h$	39	27·42 ± 0·38	2·39 ± 0·27	8·72	27·2 (51)	28·9 (29)	26·6 (41)
$c_r h$	40	64·80 ± 1·03	6·51 ± 0·73	10·05	65·7 (48)	66·6 (28)	65·0 (38)
rl	42	63·64 ± 0·99	6·39 ± 0·70	10·04	64·0 (45)	65·5 (26)	64·7 (38)
$\angle M$	42	120°·11 ± 1·15	7°·43 ± 0·81	—	120°·3 (47)	119°·7 (30)	120°·9 (39)
$\angle R$	39	71°·01 ± 1·21	7°·57 ± 0·86	—	72°·0 (36)	72°·1 (22)	68°·7 (37)
$\angle C'$	40	66°·74 ± 1·02	6°·44 ± 0·72	—	68°·2 (32)	68°·4 (28)	69°·0 (36)
$100\, c_r h/ml$	40	61·66 ± 1·09	6·88 ± 0·77	—	60·9 (27)	62·9 (25)	59·3 (31)
$100\, c_r c_r/ml$	37	94·04 ± 1·15	7·02 ± 0·82	—	94·4 (15)	92·5 (24)	93·5 (25)
$100\, g_o g_o/c_p l$	41	127·19 ± 1·76	11·25 ± 1·24	—	129·0 (32)	122·5 (27)	129·7 (38)
$100\, g_o g_o/c_r c_r$	36	100·11 ± 1·21	7·29 ± 0·86	—	99·3 (19)	97·6 (23)	100·0 (27)
$100\, rb'/rl$	42	50·72 ± 0·88	5·73 ± 0·63	—	51·5 (45)	49·4 (26)	51·0 (38)

means of the corresponding characters in the mandibles of the Anglo-Saxon skulls in the London Museums which were measured by G. M. Morant* and of the Dunstable mandibles† which were known to be very similar to this series of Anglo-Saxon mandibles are also tabulated. For the mandibular characters of the Burwell group of males, the standard deviations and coefficients of variation are also given, as the numbers of observations on which these are based are usually about 40. The standard errors of the means and standard deviations are also included in the table. For female mandibles, the means of the characters and the numbers of observations on which these are based are only given for the three groups from Burwell, Bidford-on-Avon and the London Museums. The small number of female mandibles from Dunstable did not seem to warrant even the calculation of mean values.

The means of the corresponding characters for males in the several groups can be compared in detail in Table VI, but the characters amongst a selected series in which the male mandibles of one group differ most strikingly from those of the other groups can be seen more readily by reference to Table VIII in which the values of α between the characters for each pair of groups are given. The selected

TABLE VII.

Comparative Table of Means for Mandibles of Female Skulls.

Character	Burwell	London Museums	Bidford-on-Avon
	Mean	Mean	Mean
w_1	115·9 (16)	116·6 (22)	114·1 (11)
$g_o g_o$	91·7 (19)	92·9 (35)	90·8 (16)
$c_r c_r$	92·9 (16)	93·2 (28)	94·0 (16)
zz	44·1 (20)	44·1 (50)	46·1 (19)
$c_y l$	19·2 (19)	19·1 (35)	20·4 (13)
ml	99·9 (20)	104·2 (45)	102·8 (17)
$c_p l$	73·4 (20)	74·6 (49)	76·8 (18)
rb'	30·6 (20)	31·0 (56)	31·7 (19)
$m_2 p_1$	27·9 (18)	27·6 (57)	27·8 (18)
h_1	29·2 (18)	30·5 (31)	29·4 (19)
$m_2 h$	25·3 (18)	24·4 (52)	26·8 (18)
$c_r h$	58·4 (18)	59·2 (47)	62·0 (18)
rl	57·3 (19)	59·1 (45)	61·5 (15)
$\angle M$	122°·6 (20)	122°·5 (49)	119°·9 (18)
$\angle R$	69°·5 (16)	68°·2 (36)	69°·5 (14)
$\angle C'$	67°·8 (19)	70°·6 (32)	69°·8 (17)
$100\, c_r h/ml$	58·5 (18)	58·3 (38)	60·3 (17)
$100\, c_r c_r/ml$	93·4 (16)	91·7 (26)	90·2 (14)
$100\, g_o g_o/c_p l$	125·7 (19	126·2 (35)	118·5 (16)
$100\, g_o g_o/c_r c_r$	99·2 (15)	99·3 (23)	97·1 (14)
$100\, rb'/rl$	53·8 (19)	53·0 (43)	52·1 (15)

* *Biometrika*, Vol. XVIII. (1926), p. 96.

† *Ibid.*, Vol. XXV. (1933), pp. 147—157.

TABLE VIII.

Values of $\alpha = \frac{n_s n_{s'}}{n_s + n_{s'}} \left(\frac{M_s - M_{s'}}{\sigma_s} \right)^2$ *between the Several Male Mandibular Series.*

Character	Burwell and London Museums	Burwell and Bidford-on-Avon	London Museums and Bidford-on-Avon	Burwell and Dunstable	London Museums and Dunstable	Bidford-on-Avon and Dunstable
w_1	0·75	8·57	12·20	0·95	2·98	3·93
zz	0·85	0·22	1·76	0·47	0·03	1·20
rl	0·14	2·42	1·54	1·01	0·42	0·41
rb'	5·40	0·79	1·30	1·85	0·71	0·13
$c_y l$	1·33	1·85	5·40	0·05	0·81	2·33
ml	1·89	0·17	0·74	1·12	0·11	0·31
h_1	4·30	0·48	1·55	1·39	0·78	0·17
$m_2 p_1$	1·26	0·22	0·26	1·63	0·08	0·50
$\angle R$	0°·28	0°·26	0°·00	1°·51	2°·99	2°·40
$100\, g_o g_o / c_p l$	0·56	3·44	5·91	1·18	0·08	7·82

list of mandibular characters in Table VIII is that used by G. M. Morant for the calculation of the coefficients of racial likeness between mandibular groups and we are indebted to him for giving it to us. These characters have been selected by this author because of their relatively slight correlation with one another and because of the relative frequency with which they differ significantly in a series of racial types that were studied by him.

8. *Values of* α *for a selected Series of Mandibular Characters showing the Features in which Differences in the Mandibular Groups are most emphasised**. For the Burwell and London Museums series of male mandibles the only two characters of the ten tabulated in which α exceeds 2·7 are rb' and h_1. Both these characters are smaller on the average in the Burwell series than in Morant's series.

Comparing the Burwell and Bidford-on-Avon groups, two characters, w_1 and $100\, g_o g_o / c_p l$, alone show values of α exceeding 2·7. For the bicondylar width, w_1, the value of α exceeds 8. In the Bidford series this character is almost 5 mm. less on the average than in the Burwell series; the mean value of $100\, g_o g_o / c_p l$ in the Burwell group is rather less than in the mandibles from Bidford-on-Avon.

In the London Museums and Bidford groups the values of α for three characters exceed 2·7. These are w_1, $c_y l$ and $100\, g_o g_o / c_p l$. The Bidford-on-Avon mean value of w_1 is about 6 mm. less and $100\, g_o g_o / c_p l$ almost 5 units less than the corresponding values for the mandibles from the London Museums; the Bidford series thus holds in respect of these two characters a similar relationship to the Burwell and Museums groups. Reference to Table VI shows that the mean value of $c_p l$ is of much the same order in the three groups, but that the mean value of $g_o g_o$ is 4 to

* The standard deviations used in the calculation of the α's for the mandibular characters are those of the Qau Egyptian series of the IVth dynasty which Dr G. M. Morant has kindly given to us.

5 mm. less in the Bidford group than in the other two; the difference in bigonial width is thus almost wholly responsible for the difference in the index. The average condylar length, $c_y l$, is rather less in the Bidford-on-Avon mandible than in that from the London Museums.

For the Burwell and Dunstable groups, all the values of α are under 2·7.

For the London Museums and Dunstable series only two characters, w_1 and $\angle R$, have values of α exceeding 2·7, but in neither case does the value exceed 3. The mean values of these two characters in the Dunstable mandible are rather less than the corresponding values in that from the London Museums.

In the Dunstable and Bidford-on-Avon series the values of α for two of the ten characters, namely w_1 and $100\, g_o g_o / c_p l$, are in excess of 2·7. The value for the index is almost equal to 8. The mean values of these two characters are both less in the Bidford group. The relatively narrow bicondylar width, w_1, in the Bidford group has already been commented upon. It is about 3·5 mm. less on the average than in the Dunstable group; the bigonial width, $g_o g_o$, shows a deficiency of about the same order. The mean body length, $c_p l$, on the other hand, is rather greater than in the Dunstable mandible. The relatively small bigonial width is thus again mainly responsible for the divergence in the index.

The female mandibles may be dealt with more briefly. The values of α for the selected characters in the three Anglo-Saxon series are given in Table IX.

Comparing the characters in the mandibles from Burwell with those from the London Museums, only one value of α, that for the mandibular length, ml, appreciably exceeds 2·7; the value is actually higher than 10. This character is definitely smaller in the Burwell group. The value of α for the rameal length, rl, just exceeds 2·7; in this case also the Burwell measurement is the smaller.

TABLE IX.

Values of $\alpha = \dfrac{n_s n_{s'}}{n_s + n_{s'}} \left(\dfrac{M_s - M_{s'}}{\sigma_s} \right)^2$ *between the Several Female Mandibular Series.*

Character	Burwell and London Museums	Burwell and Bidford-on-Avon	London Museums and Bidford-on-Avon
w_1	0·16	0·73	1·58
zz	0·00	5·72	8·09
rl	3·01	10·27	4·50
rb'	0·46	2·31	1·36
$c_y l$	0·07	6·18	8·90
ml	10·62	3·21	1·00
h_1	2·60	0·05	1·93
$m_2 p_1$	0·87	0·06	0·39
$\angle R$	0°·52	0°·00	0°·47
$100\, g_o g_o / c_p l$	0·03	3·81	5·51

Comparing the Burwell and Bidford-on-Avon groups, five of the ten characters have values of α exceeding 2·7. These are *zz*, *rl*, $c_y l$, *ml*, and $100\, g_o g_o / c_p l$. All these dimensions except the last are smaller in the Burwell mandible. The index, $100\, g_o g_o / c_p l$ is definitely greater in the Burwell group and the divergence seems to depend on a relatively smaller body length, $c_p l$.

Comparing the mandibles from the London Museums and Bidford-on-Avon, four of the ten characters under review have values of α in excess of 2·7. These are *zz*, *rl*, $c_y l$, and $100\, g_o g_o / c_p l$. All of these characters except the last are smaller in the mandible from the Museums; $100\, g_o g_o / c_p l$ is greater. The relationship of the mandible from the Museums to that from Bidford in respect of these four characters is thus the same as that of the Burwell mandible. The relatively greater index, $100\, g_o g_o / c_p l$, in the Museums mandibles than the Bidford appears to depend about equally on its two components as the mandibular body length, $c_p l$, in the Museums series is as much in defect of the Bidford-on-Avon value as the bigonial width, $g_o g_o$, is in excess of it.

9. *Coefficients of Racial Likeness for the Mandibular Series.* Having described the characters amongst those selected for computation of the coefficient of racial likeness in the mandibles in which differences between the groups were most apparent, attention may now be directed to the coefficients of racial likeness, crude and reduced, for the groups of male mandibles in Table X.

TABLE X.

Coefficients of Racial Likeness between the Several Male Mandibular Groups.

Crude Coefficients

	Burwell (40·1)	London Museums (42·4)	Bidford-on-Avon (26·4)	Dunstable (37·3)
Burwell (40·1) ...	—	0·68 ± ·30	0·84 ± ·30	0·12 ± ·30
London Museums (42·4)	0·68 ± ·30	—	2·07 ± ·30	−0·10 ± ·30
Bidford-on-Avon (26·4)	0·84 ± ·30	2·07 ± ·30	—	0·92 ± ·30
Dunstable (37·3) ...	0·12 ± ·30	−0·10 ± ·30	0·92 ± ·30	—

Reduced Coefficients

Burwell (40·1) ...	—	1·65 ± ·73	2·64 ± ·94	0·31 ± ·78
London Museums (42·4)	1·65 ± ·73	—	6·36 ± ·92	−0·25 ± ·76
Bidford-on-Avon (26·4)	2·64 ± ·94	6·36 ± ·92	—	2·98 ± ·97
Dunstable (37·3) ...	0·31 ± ·78	−0·25 ± ·76	2·98 ± ·97	—

The values of the crude coefficients seem to indicate that the Burwell male mandibles, at least so far as this group of selected characters is concerned, cannot be said to differ from those in the London Museums or from Dunstable, but may show a slight divergence from the Bidford-on-Avon mandibles. The last coefficient

of racial likeness almost attains the conventional level of significance. The type of mandible in the London Museums is identical with that from Dunstable, but diverges appreciably from that found at Bidford-on-Avon. The Bidford-on-Avon mandibles show a sensible divergence from those excavated at Dunstable.

The reduced coefficients of racial likeness which are also given in Table X indicate the relative degrees of relationship between the different groups. The coefficient between the Burwell mandibles and those from the London Museums is almost five times that shown between the Burwell and Dunstable groups, but this difference in value cannot be said to connote any real difference on the data available; both coefficients are statistically insignificant. The coefficient between the Burwell and Bidford-on-Avon mandibles is almost nine times that shown between the Burwell and Dunstable mandibles, and, as has been mentioned, may possibly be taken as indicative of a real divergence in the two Anglo-Saxon groups.

The coefficient between the mandibles from the London Museums and those from Bidford-on-Avon is fully twice that found between the mandibles from Burwell and the latter place. The Burwell mandibles would appear to be definitely more closely related to the Bidford-on-Avon mandibles than are those in the London Museums. The relationship of the Burwell mandible to the Bidford-on-Avon mandible is of the same order as that shown between the Bidford-on-Avon and Dunstable mandibles. The Dunstable mandibles appear to be identical in type with the Anglo-Saxon mandibles from Burwell and from the London Museums.

The coefficients of racial likeness, both crude and reduced, for the female groups of mandibles are given in Table XI. As the mean numbers of mandibles on which the coefficients are based are almost the same in the Burwell and Bidford-on-Avon groups, the respective crude coefficients between these groups and the group from the London Museums are relatively comparable without adjustment.

TABLE XI.

Coefficients of Racial Likeness between the Several Female Mandibular Groups.

	Burwell (18·5)	London Museums (41·2)	Bidford-on-Avon (16·1)
Crude Coefficients			
Burwell (18·5) ...	—	0·83 ± ·30	2·23 ± ·30
London Museums (41·2)	0·83 ± ·30	—	2·37 ± ·30
Bidford-on-Avon (16·1)	2·23 ± ·30	2·37 ± ·30	—
Reduced Coefficients			
Burwell (18·5) ...	—	3·25 ± 1·17	12·95 ± 1·74
London Museums (41·2)	3·25 ± 1·17	—	10·24 ± 1·30
Bidford-on-Avon (16·1)	12·95 ± 1·74	10·24 ± 1·30	—

The coefficient between the Burwell mandibles and the mandibles from the London Museums is less than unity and statistically insignificant, although it almost attains the conventional level of significance. That between the Bidford-on-Avon and Museums mandibles is significant and nearly three times as large as the other. The crude coefficient between the Bidford-on-Avon and Burwell groups is also significant and of about the same size as that between the mandibles from Bidford-on-Avon and the London Museums.

Adjustment of the coefficients to the values they would have, if based on 100 mandibles in the groups compared, results in a reduced coefficient between the Burwell mandibles and the mandibles from the London Museums of a value exceeding 3. The reduced coefficients between the Bidford-on-Avon mandibles on the one hand and the mandibles from Burwell and the London Museums on the other have values of about 13 and 10. These do not differ from one another to a significant degree, but are of the order of magnitude of reduced coefficients of racial likeness which are occasionally found between groups of mandibles belonging to crania of distinctly divergent racial type. On the other hand, two groups of crania may show considerable divergence in type while the corresponding groups of mandibles exhibit almost identical characters. Such was indeed found to be the case on comparing the Dunstable material with the Anglo-Saxon material in the London Museums.

10. *General Discussion and Summary.* The cranial and mandibular characters of the groups of Anglo-Saxon skulls from Burwell, Cambridgeshire, and Bidford-on-Avon, Warwickshire, have been compared both individually and collectively in considerable detail and both groups have been compared in respect of the same characters with the Anglo-Saxon skulls in the London Museums which were measured and described by G. M. Morant*. The comparison of the male crania in the two first-mentioned groups is the more important as these are almost twice as numerous as the female specimens. The main inferences to be drawn from the comparison of these male groups appear to be that the Burwell skulls, both crania and mandibles, are almost identical in type with the skulls from the London Museums. If material divergence is present in any feature, it occurs in the size of the upper face and its proportions. Both these groups of Anglo-Saxons seem to show, however, some divergence, about equivalent in degree, from those excavated at Bidford-on-Avon. Where divergence in these types is found, however, it is seen to occur mainly in those characters or features, into the accurate measurement of which the personal element largely enters and in which slight variance of technique, such as identification of terminals of diameters or arcs, on the part of different observers might probably produce average differences of the order that are found in the present case. Among such characters are the orbital width O_1, R, and, as a consequence, the orbital index $100\,O_2/O_1$, R, the palatal length G_1, the palatal breadth G_2, and the transverse arc Q'. The nasal height, NH, R or NH' is also such a character although it shows no divergence in the types under review. In

* *Biometrika*, Vol. XVIII. (1926), p. 76.

only one or two of its other characters, amongst those selected for comparison, and notably the nasal angle (N), does the Bidford cranium differ sensibly from the other two types. Comment has already been made on the relatively large mean measurement of the facial depth (GL) in the Bidford cranium as compared with that shown in the other types and the influence of this chord on the size of the angle N. The differences in the recorded mean measurements of these characters may be real but some caution is advisable in accepting them as of the order of magnitude found, when, as in the present case, the measurements in the different series have been taken by different observers. Fairly substantial differences in certain characters might probably be expected in Anglo-Saxon cranial material from different localities. Reference has been made to the probable existence of two distinct types, recognizable on general inspection, amongst the male crania from the Bidford-on-Avon site. Morant* on the other hand distributed the Anglo-Saxon skulls in the London Museums according to their place of origin into the four groups: West Saxons, South Saxons, Angles and Jutes, and came to the conclusion from a comparison of the actual mean measurements of the principal characters in these groups and the coefficients of racial likeness between them "that they represent populations which are extremely similar if not absolutely identical. Only one C.R.L.—that between the Angles and West Saxons—suggests any real difference of type."

Because of the relatively small numbers of female skulls in the Burwell and Bidford-on-Avon groups, very slight reliance can be placed on the mean values of the characters which are tabulated for these being truly representative of the types, but it is of interest to note in what respects, if any, the data available corroborate the conclusions that have been drawn from a comparison of the male series. The number of female crania from the London Museums is definitely greater and should provide more reliable means. The Burwell female, unlike the male, differs appreciably in many of its characters from that in the London Museums. We have referred to the presence of a few distinctly brachycephalic skulls in the Burwell group. These are mainly responsible for the relatively great mean maximum breadth (B) in the Burwell female and its divergence in this character and in the indices of which it is a component, $100\,B/L$ and $100\,B/H'$, from the female Anglo-Saxons in the Museums. The minimum frontal breadth (B') is also significantly greater in the Burwell female skull. Significant differences are also shown in the palatal length and breadth, the palatal index, the nasal breadth, the nasal index, the transverse arc (Q') and the profile angle ($\angle P$). The mean differences are so many and so relatively large that they suggest the female crania from the two sources are really of different type.

The Bidford-on-Avon female cranium differs from that of the London Museums in its orbital, nasal and palatal measurements, the characters in which differences may most readily arise from slight differences in technique as has been mentioned in the comparison of the male skulls. In their general form, these two female

* *Biometrika*, Vol. XVIII. (1926), p. 76.

crania would appear to be almost identical, but when all (the selected) characters are considered, they seem to show a divergence in type of about the same degree as that found in the corresponding male groups.

From the Burwell female cranium, the Bidford-on-Avon cranium differs mainly in having a relatively smaller maximum parietal breadth (B) with, as a consequence, smaller length-breadth ($100\,B/L$) and height-breadth ($100\,B/H'$) indices; a greater basinasal length and a smaller profile angle ($\angle P$). When all the shape characters (indices and angles) are considered, the two crania show a divergence in type, and when the comparison embraces all the selected characters some real divergence in type still seems to be indicated. The relatively small profile angle ($\angle P$) in the Bidford-on-Avon female and also in the female Anglo-Saxon from the London Museums as compared with that in the Burwell female, and the possibility of the existence of a relative prognathism in the Anglo-Saxon female, although it is not revealed in the specimens from Burwell, has already been discussed.

The data for the female Anglo-Saxon crania thus seem to confirm the existence of some divergence in type between the Bidford-on-Avon and Burwell, and Bidford-on-Avon and Museums groups which was suggested by a comparison of the corresponding male groups. Comparison of the female crania from Burwell and the London Museums would also suggest some diversity in these types rather than the identity which seemed to be established by a study of the characters in the male groups, but little importance can be attached to the apparent diversity of type in the female groups in view of the heterogeneity known to exist in the female crania from Burwell and their small number.

Turning to the mandibular relationships, there seems to be evidence that, so far as the selected characters are concerned, the Burwell male mandible does not differ significantly from the Anglo-Saxon mandible in the London Museums or from that of the specimens excavated at Bidford-on-Avon, although in the latter case the coefficient of racial likeness almost attains the conventional level of significance (three times p.e.). The male mandible for the London Museums, on the other hand, differs sensibly in the selected features from that found at Bidford-on-Avon.

While the Burwell female mandible cannot be said to differ certainly from the female Anglo-Saxon mandible in the London Museums, although the coefficient of racial likeness between them is on the border line of significance, both the female Anglo-Saxon mandible from Burwell and that described by Morant appear to differ significantly from the Bidford-on-Avon type. The reduced coefficients of racial likeness in the last two cases are as large as are those occasionally found between samples of mandibles that belong to crania of quite diverse racial type.

In a general way it may be stated that the study of the mandibles confirms largely the evidence obtained from the crania as to the interrelationships of the three groups of Anglo-Saxon skulls.

Serial Letter or No.	*F*	*L*	*B*	*B′*	*H′*	*OH*	*LB*	*Q*	*Q′*	*S*	S_1	S_2	S_3	S_1'	S_2'	S_3'
X 2	187·0	187·0	138·0	92·0	137·5	112·5	109·0	305·0	306·0	371·0	127·0	125·0	119·0	111·5	115·0	—
X 5	187·0	189·0	137·0	96·0	—	—	—	—	—	364·0	127·0	130·0	107·0	112·5	119·0	86·0
X 8	182·0	185·0	139·0	95·0	—	—	—	—	—	364·0	130·0	122·0	112·0	113·0	110·0	95·0
10	203·0	205·0	155·0	98·0	144·0	121·5	110·0	335·0	338·0	404·0	133·0	139·0	132·0	120·0	127·0	107·0
11	188·0	192·0	132·0	92·0	—	—	—	—	—	383·0	135·0	129·0	119·0	119·0	118·0	98·0
15	193·0	193·0	135·0	96·0	128·0	113·0	95·0	—	—	384·0	132·0	130·0	122·0	109·0	119·0	97·0
16	183·0	189·0	131·0	91·5	130·0	110·5	99·0	294·0	298·0	370·0	113·0	132·0	125·0	105·0	118·0	101·0
22	196·0	198·0	136·0	96·0	139·0	—	113·0	—	—	378·0	123·0	127·0	128·0	110·0	115·0	103·5
23	194·0	195·0	145·0	99·5	135·0	113·0	103·0	309·0	312·0	387·0	136·0	125·0	126·0	118·0	114·0	110·0
27 i	189·0	196·5	138·0	90·0	130·0	111·5	105·0	305·0	307·0	375·0	116·0	127·0	132·0	105·0	117·0	100·5
28	—	194·0	141·5	—	141·0	112·5	101·0	305·0	308·0	386·0	130·0	122·0	134·0	113·5	110·5	104·0
29	187·5	187·0	154·0	92·5	135·0	118·5	93·0	327·0	335·0	388·0	132·0	128·0	128·0	118·0	115·0	97·5
31	189·0	193·0	142·0	93·5	141·0	120·5	101·0	323·0	326·0	394·0	132·0	142·0	120·0	117·0	125·0	102·0
34	180·5	181·0	142·5	94·0	145·0	118·0	104·5	322·0	323·0	377·0	129·0	134·0	114·0	111·0	118·0	99·0
37	176·0	182·0	136·0	86·0	135·0	108·5	100·5	295·0	298·0	361·0	114·0	134·0	113·0	102·0	118·0	94·0
42	185·0	185·5	139·0	95·0	128·0	112·0	96·0	306·0	311·0	371·0	128·0	128·0	115·0	112·0	114·5	97·0
43	193·0	192·0	146·0	98·5	145·0	118·5	103·0	320·0	322·0	392·0	137·0	132·0	123·0	117·0	123·0	97·5
44	190·5	192·5	145·0	94·0	—	122·5	105·5	326·0	331·0	390·0	132·0	129·0	129·0	114·5	116·0	97·0
47	183·5	189·0	140·0	87·0 ?	140·0	113·5	113·5	306·0	309·0	—	122·0	125·0	—	109·0	113·0	—
56	196·5	201·0	149·5	97·5	141·0	115·5	108·0	311·0	318·0	402·0	137·0	139·0	126·0	121·5	128·0	100·5
58	192·0	196·5	145·5	98·5	137·0	114·5	102·0	313·0	316·0	394·0	130·0	141·0	123·0	112·5	123·0	103·0
62	—	—	152·5	95·0	—	—	—	—	—	—	132·0	137·0	—	117·0	124·0	—
66	182·0	184·0	143·0	94·0	128·0	114·5	97·0	312·0	317·0	372·0	129·0	130·0	113·0	114·0	115·0	94·0
67	194·5	197·0	139·0	98·0	133·0	118·5	105·0	322·0	326·0	382·0	124·0	134·0	124·0	110·0	122·0	100·0
68	184·0	185·0	133·0	—	138·0	111·5	101·0	308·0	308·0 ?	377·0	127·0	120·0	130·0	110·0	108·0	112·0
70	188·0	188·5	137·5	99·0	131·5	107·0	100·5	298·0	—	374·0	126·0	112·0	136·0	110·0	105·5	107·0
74	185·0	185·0	139·0	96·0	130·5	113·5	100·5	305·0	308·0	371·0	128·0	122·0	121·0	103·0	108·5	98·0
77	182·5	184·0	148·0	105·0	138·0	127·0	104·0	338·0	342·0	380·0	124·0	138·0	118·0	109·0	117·0	98·0
78	183·5	186·5	146·5	98·5	126·0	111·0	96·0	306·0	311·0	379·0	122·0	136·0	121·0	110·0	119·0	94·0
89	193·0	197·5	141·5	94·0	142·5	120·0	109·0	322·0	327·0	396·0	133·0	131·0	132·0	116·0	117·0	107·5
92	184·5	188·5	143·0	94·0	135·0	117·5	102·0	318·0	319·0	385·0	127·0	138·0	120·0	110·0	120·0	100·0
93	186·0	189·0	143·0	95·0	142·0	120·0	102·0	319·0	322·0	387·0	135·0	131·0	121·0	118·0	117·0	101·0
98	180·0	180·0	142·0	93·0	128·0	112·5	91·0	322·0	323·0	374·0	129·0	136·0	109·0	109·0	121·0	95·0
100	184·0	183·0	146·5	94·5	129·0	—	95·0	—	—	376·0	126·0	128·0	122·0	110·0	115·0	98·0
102	185·0	185·5	142·0	94·0	131·0	111·5	—	311·0	316·0	390·0	143·0	121·0	126·0	119·0	109·5	103·5
106	183·5	185·0	134·0	95·0	136·5	108·0	99·5	300·0	306·0	377·0	131·0	123·0	123·0	109·0	111·0	102·0
107	190·0	193·5	143·0	98·0	134·0	114·5	103·0	309·0	315·0	394·0	125·0	151·0	118·0	113·0	133·0	94·0
108	187·0	188·0	140·0	96·5	135·5	—	95·0	—	—	385·0	130·0	129·0	116·0	116·5	120·0	99·0
111	187·0	190·0	142·0	92·5	146·5	116·0	108·0	321·0	326·0	386·0	126·0	136·0	124·0	111·5	123·0	104·0
112	179·0	180·0	140·0	94·0	140·0	118·5	104·0	318·0	324·0	373·0	118·0	135·0	120·0	106·0	115·0	92·0
115	188·0	192·0	137·0	98·0	137·0	109·5	103·0	302·0	307·0	385·0	132·0	133·0	120·0	114·0	117·0	101·0
121	188·0	189·0	146·5	95·0	130·0	107·5	100·0	301·0	306·0	380·0	132·0	128·0	120·0	117·0	115·0	97·0
122	188·0	191·0	145·0	96·0	131·0	107·5	102·0	299·0	303·0	368·0	127·0	120·0	121·0	112·5	108·5	98·0
123	186·0	189·0	147·0	95·0	142·0	111·5	106·0	308·0	312·0	378·0	126·0	140·0	112·0	112·0	123·0	92·5
133	183·5	184·5	137·0	95·0	—	—	—	—	—	—	—	—	—	—	—	—
140	189·0	194·0	153·0	102·0	146·0	122·0	109·5	335·0	339·0	397·0	126·0	134·0	137·0	112·0	121·0	114·0

Serial Letter or No.	*F*	*L*	*B*	*B′*	*H′*	*OH*	*LB*	*Q*	*Q′*	*S*	S_1	S_2	S_3	S_1'	S_2'	S_3'
X 1	180·0	181·0	132·5	98·0	—	—	—	—	—	—	120·0	121·0	—	106·5	111·5	—
2 *a*	180·0	179·0	137·0	99·0	—	—	—	—	—	353·0	119·0	125·0	109·0	103·5	114·0	92·0
2 *b*	178·0	180·0	139·0	89·5	—	—	—	—	—	373·0	122·0	127·0	123·0	107·0	115·0	101·0
13	192·0	191·5	133·5	93·0	137·0	115·0	106·0	304·0	308·0	385·0	125·0	138·0	122·0	111·0	124·0	96·0
24	—	183·0	132·0	—	—	—	—	—	—	363·0	125·0	118·0	120·0	111·0	108·0	101·0
25	183·0	184·0	133·0	95·0	129·0	111·5	99·0	300·0	303·0	369·0	126·0	125·0	118·0	111·0	116·0	96·0
26	181·5	183·5	133·0	95·0	119·0	—	94·5	—	—	355·0	123·0	117·0	115·0	110·0	108·0	92·0
30	184·0	183·0	—	—	—	114·5	—	311·0	314·0	361·0	121·0	125·0	115·0	104·5	111·5	95·0
32	174·0	173·0	142·0	94·0	129·0	112·5	83·0	305·0	311·0	374·0	119·0	131·0	124·0	101·5	115·0	102·0
36	178·0	180·5	—	101·0	—	—	—	—	—	357·0	127·0	124·0	106·0	114·0	112·0	93·0
39	—	—	—	—	138·5 ?	—	102·0	—	—	338·0	124·0	—	—	104·5	—	—
40	180·5	182·0	143·0	96·0	125·0	112·5	98·0	305·0	311·0	365·0	129·0	121·0	115·0	110·0	110·0	95·0
52	176·0	179·0	135·5	97·0	123·0	107·0	94·0	288·5	292·0	352·0	114·0	128·0	110·0	101·5	114·5	93·0
53	173·0	173·0	145·0	—	133·0	118·5	87·0	327·0	327·0	370·0	125·0	125·0	120·0	110·0	108·5	99·0
57	187·0	188·0	145·0	108·0 ?	—	—	—	—	—	384·0	142·0	127·0	115·0	124·0	114·5	95·0
59	190·5	191·0	141·0	92·0	127·0	110·5	96·0	301·0	307·0	380·0	127·0	125·0	128·0	112·0	115·0	102·0
61	182·0	184·5	139·5	92·0	122·0	106·5	89·0	296·0	299·0	380·0	134·0	131·0	115·0	118·0	115·0	92·0
80	193·0	193·0	136·0	97·0	—	115·5	—	311·0	318·0	366·0	125·0	125·0	116·0	109·0	113·0	95·0
94	179·0	179·0	149·0	95·0	133·0	121·5	95·0	331·0	340·0	—	—	—	—	—	120·0	—
97	178·0	180·0	130·0	94·0	128·5	100·0	99·0	281·0	287·0	349·0	116·0	131·0	102·0	103·0	118·5	88·0
119	185·0	186·0	140·0	—	—	—	—	—	—	—	—	—	—	—	—	—
127	181·0	183·0	140·0	95·0	—	107·5	—	298·0 ?	304·0 ?	—	—	—	—	—	—	—
135	187·0	186·0	145·5	95·0	137·0	110·0	93·0	312·0	315·0	390·0	135·0	130·0	125·0	114·0	117·0	103·5

APPENDIX V. INDIVIDUAL MEASUREMENTS OF MALE ANGLO-S

S_3'	S_3'	U	PH	GH	$G'H$	GB	J	NH'	NH,R	NH,L	NB	DC	DA	SC	O_1',R	O_1,R	O_1,L	O_2,R
115·0	—	520·0	17·5 ?	120·5	70·0	93·5	—	53·0	53·0	54·0	26·0	—	—	—	—	—	41·0 ?	—
119·0	86·0	524·0	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
110·0	95·0	511·0	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
127·0	107·0	568·0	—	124·0	—	88·0	141·0 ?	—	—	—	—	—	—	—	—	—	42·0	—
118·0	98·0	522·0 ?	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
119·0	97·0	531·0	—	—	—	—	—	—	—	—	—	—	—	7·0	—	—	41·0	—
118·0	101·0	514·0	—	111·0	66·0	89·0	127·0	47·0	49·0	49·0	23·5	—	—	—	—	41·0	41·0	32·0
115·0	103·5	534·0	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
114·0	110·0	542·0	—	119·0	71·0	104·0	141·0	53·0	55·0	56·0	27·0	24·5	40·0 ?	9·0	40·0	44·0	44·0	33·0
117·0	100·5	521·0	28·5 ?	115·0	73·0	94·0	135·0	49·5	50·0	49·5	23·0	21·0	41·0	9·0	42·0	43·0	43·0	30·0
110·5	104·0	—	—	—	67·0	—	—	44·0	44·0	43·5	22·0	—	—	—	—	40·0	—	36·0
115·0	97·5	532·0	—	106·0	63·5	93·0	—	44·0	44·0	44·5	21·0	19·0	36·0 ?	9·0	38·5	40·5	40·5	32·0
125·0	102·0	532·0	—	122·0	72·0	100·5	139·0	51·5	50·0	51·5	23·0	20·5	30·0 ?	4·5	42·5	43·0	43·0	30·0
118·0	99·0	515·0	—	120·0	72·0	99·0	—	57·0	53·5	55·0	22·5	—	—	—	—	—	—	—
118·0	94·0	502·0	—	112·0	65·0	82·0	—	47·0	48·0	46·0	24·5	—	—	—	—	40·0	43·0	33·0
114·5	97·0	516·0	24·0	115·0	68·0	94·0	—	45·0	47·0	45·0	—	—	—	—	—	43·0	41·0	—
123·0	97·5	540·0	21·0	124·5	75·0	87·5	—	54·5	53·0	54·5	—	—	—	—	—	—	—	—
116·0	97·0	534·0	—	107·0	60·0	87·5	—	48·0	46·5	46·5	23·0	—	—	9·0	—	39·0	39·0	30·0
113·0	—	518·0	19·0	114·5	67·0	97·0	—	49·0	47·0	49·5	25·0	—	—	—	—	—	41·0	—
128·0	100·5	553·0	—	116·0	73·0	104·0	148·5	52·0	52·0	52·0	27·0	26·0	42·0	10·0	37·0	42·0	41·0	34·0
123·0	103·0	540·0	20·9	111·5	70·0	98·5	135·0	50·5	51·0	52·0	24·0	25·5	36·0	14·5	37·0	40·0	40·0	33·0
124·0	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
115·0	94·0	520·0	—	115·5	—	93·0	131·0	50·0	50·0	49·0	23·0	—	—	8·0	—	40·0	39·5	32·0
122·0	100·0	538·0	—	123·0 ?	—	110·0	—	51·5	52·0	52·0	25·0	—	—	11·0	—	43·0	44·0	35·0
108·0	112·0	—	—	112·0	68·5	—	—	52·0	53·0	52·0	27·0	—	—	—	—	—	—	—
105·5	107·0	524·0	—	123·0	75·0	93·0	125·0	53·0	53·0	52·5	25·0	24·0 ?	37·0 ?	9·0	41·5	44·5	43·5	32·5
108·5	98·0	511·0	—	123·5	—	96·0	133·0	52·0	51·0	50·5	23·5	22·5 ?	39·0	10·5	41·0	42·0	42·0	33·5
117·0	98·0	538·0	24·0 ?	122·5	75·0	95·0	137·0 ?	52·0	51·5	51·5	22·0	—	—	8·0	—	43·0	42·0	34·0
119·0	94·0	529·0	18·5	107·0	67·5	91·0	131·0	50·5	51·0	49·0	25·5	—	—	7·0	—	41·0	40·5	32·0
117·0	107·5	540·0	—	125·0	79·0	92·0	130·0	57·0 ?	57·0 ?	56·0 ?	20·0 ?	—	—	—	—	—	43·0	35·0
120·0	100·0	527·0	17·0	104·5	61·0	93·5	138·0	46·0	46·0	47·0	26·0	—	—	9·5	—	43·5	43·0	30·0
117·0	101·0	525·0	18·0	110·5	66·0	99·0	—	49·0	49·0	48·0	22·0	21·0	38·0	7·0	38·5	41·0	40·0	34·5
121·0	95·0	519·0	20·0	111·0	67·0	92·0	125·5	48·5	49·0	49·5	22·0	—	—	8·0	—	41·0	40·5	32·5
115·0	98·0	524·0	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
109·5	103·5	522·0	16·0	103·0	58·0	93·0	130·0	44·0	44·0	43·0	26·0	—	—	—	—	43·0	42·0	36·0
111·0	102·0	513·0	18·0	—	61·5	87·0	124·0	44·0	42·0	43·0	22·0	—	—	—	—	39·0	41·0	35·0
133·0	94·0	536·0	22·5	119·0	74·0	—	—	51·5	53·0	52·0	25·0	25·0	40·0	12·0	37·0	42·0	42·0	33·0
120·0	99·0	520·0	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
123·0	104·0	534·0	18·5	118·5	69·0	106·0	144·0	51·0	52·0	51·0	25·0	20·0	39·0	8·0	42·0	45·0	45·0	33·5
115·0	92·0	506·0	20·5	115·0	69·0	96·0	128·0	47·0	47·0	46·0	24·0	—	—	7·0	—	39·0	38·0	27·0
117·0	101·0	523·0	18·0	115·0	70·0	97·5	133·0	51·0	53·0	51·5	25·0	—	—	10·0	—	44·0	43·0	32·5
115·0	97·0	533·0	—	110·0	64·0	96·0	131·0	48·0	47·0	48·0	25·0	22·5	36·0	9·0	39·0	42·0	42·0	32·0
108·5	98·0	527·0	16·0	116·5	71·0	96·5	134·0	56·0	55·5	56·0	24·0	—	—	6·0	—	46·0	46·0	39·0
123·0	92·5	536·0	16·0	112·0	67·0	96·5	132·5	51·0	51·0	51·0	25·0	—	—	9·0	—	44·0	42·0	32·5
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
121·0	114·0	552·0	—	130·0	77·0	100·0	143·0	54·5	54·0	54·0	26·5	—	—	11·0	—	42·0	42·0	34·0

APPENDIX VI. INDIVIDUAL MEASUREMENTS OF FEMALE ANGLO-SA

S_3'	S_3'	U	PH	GH	$G'H$	GB	J	NH'	NH,R	NH,L	NB	DC	DA	SC	O_1',R	O_1,R	O_1,L	O_2,R
111·5	—	510·0	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
114·0	92·0	513·0	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
115·0	101·0	505·0	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
124·0	96·0	522·0	—	—	—	83·0	—	54·0	55·0	54·0	23·0	—	—	—	—	—	—	—
108·0	101·0	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
116·0	96·0	507·0	17·0	114·0	65·0	86·0	126·0	47·5	47·0	47·5	21·0	—	—	10·0	—	43·0	41·0	32·0
108·0	92·0	507·0	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
111·5	95·0	—	—	—	64·0	83·5	—	45·5	45·0	45·5	22·0	—	—	10·0 ?	—	—	39·5	—
115·0	102·0	503·0	18·0	115·0	62·5	85·0	—	45·0	45·0	44·5	22·0	—	—	9·0	—	40·0	40·0	32·5
112·0	93·0	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
110·0	95·0	518·0 ?	17·0	107·0	61·0	91·0	128·0	44·5	46·0	45·0	22·0	—	—	8·5	—	40·0	39·0	31·5
114·5	93·0	498·0	18·0	—	60·0	95·0	128·0	43·5	44·0	43·5	24·0	—	—	9·0	—	41·0	41·0	33·0
108·5	99·0	503·0	—	109·0	68·0	85·5	—	50·0	50·0	50·0	23·0	—	—	—	—	—	—	—
114·5	95·0	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
115·0	102·0	524·0	—	115·0	72·5	99·0	129·0	50·0	50·0	51·0	23·5	20·5	37·0	9·0	36·5	40·0	40·0	34·0
115·0	92·0	516·0	20·0	—	66·0	85·0	121·0	48·0	49·0	49·5	27·0	22·0	32·0	6·5	33·0	40·0	39·0	34·0
113·0	95·0	528·0	—	—	70·0	83·0	—	—	—	—	—	—	—	8·0	—	42·0	—	35·0
120·0	—	—	17·5	—	65·0	84·0	—	48·0	47·5	47·0	21·0	—	—	—	—	—	—	—
118·5	88·0	496·0	19·0	106·5 ?	68·0	91·0	—	49·0	47·0	48·0	—	—	—	—	—	—	—	—
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
117·0	103·5	525·0	20·0	102·0	62·0	82·5	—	42·0	42·0	42·0	22·0	—	—	—	—	42·0	—	34·0

MEASUREMENTS OF MALE ANGLO-SAXON CRANIA FROM BURWELL.

NH, L	*NB*	*DC*	*DA*	*SC*	O_1', R	O_1, R	O_1, L	O_2, R	O_2, L	*EOW*	G_1	G_1'	G_2	*GL*	*fml*	*fmb*	100 *B/L*	100 *B/F*
54·0	26·0	—	—	—	—	—	41·0 ?	—	33·0 ?	97·0	—	—	35·0	98·0	—	33·0	73·8	73·8
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	39·0	28·0 ?	72·5	73·3
—	—	—	—	—	—	—	—	—	—	107·0	—	—	—	—	—	—	75·1	76·4
—	—	—	—	—	—	—	42·0	—	32·0	103·0	—	—	—	—	41·0	31·0	75·6	76·4
—	—	—	—	—	—	—	—	—	—	92·0	—	—	—	—	—	—	68·8	71·0
—	—	—	—	7·0	—	—	41·0	—	33·0	96·5	—	—	—	—	39·0	31·0	70·0	70·0
49·0	23·5	—	—	—	—	41·0	41·0	32·0	32·0	96·0	—	—	33·0 ?	95·0	39·0	—	69·3	71·6
—	—	—	—	—	—	—	—	—	—	99·0	—	—	—	—	43·5	—	68·7	69·4
56·0	27·0	24·5	40·0 ?	9·0	40·0	44·0	44·0	33·0	34·0	102·0	—	—	—	98·0	49·5	32·0	74·4	74·7
49·5	23·0	21·0	41·0	9·0	42·0	43·0	43·0	30·0	31·0	102·0	46·0	42·0	41·0	90·0	35·0	30·0	70·2	73·0
43·5	22·0	—	—	—	—	40·0	—	36·0	—	—	—	—	36·0	96·0	41·5	33·0	72·9	—
44·5	21·0	19·0	36·0 ?	9·0	38·5	40·5	40·5	32·0	32·5	96·0	—	39·5	36·5	87·0	35·0	32·0	82·4	82·1
51·5	23·0	20·5	30·0 ?	4·5	42·5	43·0	43·0	30·0	30·0	100·5	—	—	45·5	98·0	39·0	33·0	73·6	75·1
55·0	22·5	—	—	—	—	—	—	—	—	95·0	—	—	41·0	104·0	36·0	35·5	78·7	78·9
46·0	24·5	—	—	—	—	40·0	43·0	33·0	34·0	98·0	—	—	—	91·5	35·0	30·0	74·7	77·3
45·0	—	—	—	—	—	43·0	41·0	—	—	96·0	—	46·0	38·0	91·0	41·0	32·0	74·9	75·1
54·5	—	—	—	—	—	—	—	—	—	93·0	42·5	40·0	37·0	89·0	38·0	34·0	76·0	75·7
46·5	23·0	—	—	9·0	—	39·0	39·0	30·0	30·0	95·0	—	—	—	99·5	41·0	32·0	75·3	76·1
49·5	25·0	—	—	—	—	—	41·0	—	30·0	—	—	—	42·5 ?	112·0	—	—	74·1	76·3
52·0	27·0	26·0	42·0	10·0	37·0	42·0	41·0	34·0	33·0	101·0	—	—	42·0 ?	96·0	36·0	31·0	74·4	76·1
52·0	24·0	25·5	36·0.	14·5	37·0	40·0	40·0	33·0	33·0	96·0	47·0	44·0	38·0	91·0	39·0	31·0	74·0	75·8
—	—	—	—	—	—	—	—	—	—	103·5	—	—	—	—	—	—	—	—
49·0	23·0	—	—	8·0	—	40·0	39·5	32·0	32·0	94·0	—	—	40·0	—	33·0	26·0	77·7	78·6
52·0	25·0	—	—	11·0	—	43·0	44·0	35·0	36·0	—	—	—	—	—	41·0	26·5	70·6	71·5
52·0	27·0	—	—	—	—	—	—	—	—	—	—	—	44·0	93·0	34·0	31·0	71·9	72·3
52·5	25·0	24·0 ?	37·0 ?	9·0	41·5	44·5	43·5	32·5	33·5	99·5	54·0	50·0	40·0	95·0	37·0	28·5	72·9	73·1
50·5	23·5	22·5 ?	39·0	10·5	41·0	42·0	42·0	33·5	32·5	101·0	—	—	42·0	—	33·0	31·0	75·1	75·1
51·5	22·0	—	—	8·0	—	43·0	42·0	34·0	32·5	104·0	51·0	46·0	41·0	96·0	34·0	—	80·4	81·1
49·0	25·5	—	—	7·0	—	41·0	40·5	32·0	32·0	94·0	—	—	39·0 ?	87·0	32·5	32·0	78·6	79·8
56·0 ?	20·0 ?	—	—	—	—	—	43·0	35·0	35·0	99·0	—	—	39·0	96·5	38·0	28·5	71·6	73·3
47·0	26·0	—	—	9·5	—	43·5	43·0	30·0	—	100·5	—	—	40·0	90·0	37·0	29·5	75·9	77·5
48·0	22·0	21·0	38·0	7·0	38·5	41·0	40·0	34·5	34·5	96·0	46·5	42·5	41·0	89·0	35·0	29·0	75·7	76·9
49·5	22·0	—	—	8·0	—	41·0	40·5	32·5	33·0	91·0	48·0	42·0	40·0	85·0	33·0	31·0	78·9	78·9
—	—	—	—	—	—	—	—	—	—	98·0	—	—	—	—	37·0	32·0	80·1	79·6
43·0	26·0	—	—	—	—	43·0	42·0	36·0	36·0	102·0	—	—	42·5	84·5	35·5	—	76·5	76·8
43·0	22·0	—	—	—	—	39·0	41·0	35·0	34·0	99·0	50·0	48·5	—	91·0	36·0	29·5	72·4	73·0
52·0	25·0	25·0	40·0	12·0	37·0	42·0	42·0	33·0	33·0	103·0	49·0	46·0	41·0	92·5	35·0	32·0	73·9	75·3
—	—	—	—	—	—	—	—	—	—	99·0	—	—	—	—	38·5	32·0	74·5	74·9
51·0	25·0	20·0	39·0	8·0	42·0	45·0	45·0	33·5	33·0	102·5	47·0	43·0	43·0	88·0	35·0	30·5	74·7	75·9
46·0	24·0	—	—	7·0	—	39·0	38·0	27·0	28·0	94·0	50·5	45·5	40·0	100·0	30·0	27·0	77·8	78·2
51·5	25·0	—	—	10·0	—	44·0	43·0	32·5	32·0	99·0	—	—	—	91·0	40·0	35·0	71·4	72·9
48·0	25·0	22·5	36·0	9·0	39·0	42·0	42·0	32·0	31·5	100·0	52·0	49·0	39·0	96·0	36·0	30·0	77·3	77·9
56·0	24·0	—	—	6·0	—	46·0	46·0	39·0	39·0	103·0	—	—	41·0	96·0	38·0	33·0	75·9	77·1
51·0	25·0	—	—	9·0	—	44·0	42·0	32·5	33·0	100·0	—	—	46·0	97·0	38·0	34·0	77·8	79·0
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	74·3	74·6
54·0	26·5	—	—	11·0	—	42·0	42·0	34·0	32·5	—	—	—	41·0	102·0	35·0	30·5	78·9	81·4

ASUREMENTS OF FEMALE ANGLO-SAXON CRANIA FROM BURWELL.

NH, L	*NB*	*DC*	*DA*	*SC*	O_1', R	O_1, R	O_1, L	O_2, R	O_2, L	*EOW*	G_1	G_1'	G_2	*GL*	*fml*	*fmb*	100 *B/L*	100 *B/F*
—	—	—	—	—	—	—	—	—	—	98·0	—	—	—	—	—	—	73·2	73·6
—	—	—	—	—	—	—	—	—	—	101·0	—	—	—	—	—	—	76·5	76·1
—	—	—	—	—	—	—	—	—	—	93·0 ?	—	—	—	—	—	—	77·2	78·1
54·0	23·0	—	—	—	—	—	—	—	—	90·0	—	—	36·0	—	32·0	25·0	69·7	69·5
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	72·1	—
47·5	21·0	—	—	10·0	—	43·0	41·0	32·0	32·0	96·0	—	—	38·0	91·0	37·0	31·0	72·3	72·7
—	—	—	—	—	—	—	—	—	—	100·5	—	—	—	—	39·0	29·0	72·5	73·3
45·5	22·0	—	—	10·0 ?	—	—	39·5	—	35·5	—	—	—	34·0	—	—	—	—	—
44·5	22·0	—	—	9·0	—	40·0	40·0	32·5	31·0	93·5	—	—	36·0	86·0	33·0	30·0	82·1	81·6
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	34·0	30·0	—	—
45·0	22·0	—	—	8·5	—	40·0	39·0	31·5	31·0	93·5	52·5	47·0	41·5	96·0	38·0	31·5	78·6	79·2
43·5	24·0	—	—	9·0	—	41·0	41·0	33·0	32·0	99·0	—	—	—	—	40·0	—	75·7	77·0
50·0	23·0	—	—	—	—	—	—	—	—	—	—	—	—	—	37·0	26·5	83·8	83·8
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	77·1	77·5
51·0	23·5	20·5	37·0	9·0	36·5	40·0	40·0	34·0	34·0	93·5	—	—	39·5	92·0	33·0	28·0	73·8	74·0
49·5	27·0	22·0	32·0	6·5	33·0	40·0	39·0	34·0	34·0	94·0	—	—	—	84·0	31·0	27·0	75·6	76·6
—	—	—	—	8·0	—	42·0	—	35·0	—	—	—	—	42·0	—	—	—	70·5	70·5
47·0	21·0	—	—	—	—	—	—	—	—	89·0	41·0	39·0	37·0	85·0	36·0	28·0	83·2	83·2
48·0	—	—	—	—	—	—	—	—	—	98·0	—	—	—	95·0	41·0	—	72·2	73·0
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	75·3	75·7
—	—	—	—	—	—	—	—	—	—	—	45·0	44·0	36·5	—	—	—	—	—
42·0	22·0	—	—	—	—	42·0	—	34·0	35·0	92·5	—	—	35·0	89·0	40·0	35·0	78·2	77·8

	100 B/L	100 B/F	100 H'/L	100 B/H'	100 G'H/GB	100 G'H/J	100 NB/NH'	100 NB/NH, R	100 NB/NH, L	100 O_2/O_1', R	100 O_2/O_1, R
	73·8	73·8	73·5	100·4	74·9	—	49·1	49·1	48·1	—	—
?	72·5	73·3	—	—	—	—	—	—	—	—	—
	75·1	76·4	—	—	—	—	—	—	—	—	—
	75·6	76·4	70·2	107·6	—	—	—	—	—	—	—
	68·8	71·0	—	—	—	—	—	—	—	—	—
	70·0	70·0	66·3	105·5	—	—	—	—	—	—	—
	69·3	71·6	68·8	100·8	74·2	52·0	50·0	48·0	48·0	—	78·0
	68·7	69·4	70·2	97·8	—	—	—	—	—	—	—
	74·4	74·7	69·2	107·4	68·3	50·4	50·9	49·1	48·2	82·5	75·0
	70·2	73·0	66·2	106·2	77·7	54·1	46·5	46·0	46·5	71·4	69·8
	72·9	—	72·7	100·4	—	—	50·0	50·0	50·6	—	95·0
	82·4	82·1	72·2	114·1	68·3	—	47·7	47·7	47·2	83·1	79·0
	73·6	75·1	73·1	100·7	71·6	51·8	44·7	46·0	44·7	70·6	69·8
	78·7	78·9	80·1	98·3	72·7	—	39·5	42·1	40·9	—	—
	74·7	77·3	74·2	100·7	79·3	—	52·1	51·1	53·3	—	82·5
	74·9	75·1	69·0	108·6	72·3	—	—	—	—	—	—
	76·0	75·7	75·5	100·7	85·7	—	—	—	—	—	—
	75·3	76·1	—	—	68·6	—	47·9	49·5	49·5	—	76·9
	74·1	76·3	74·1	100·0	69·1	—	51·0	53·2	50·5	—	—
	74·4	76·1	70·2	106·0	70·2	49·2	51·9	51·9	51·9	91·9	81·0
	74·0	75·8	69·7	106·2	71·1	51·9	47·5	47·1	46·2	89·2	82·5
	—	—	—	—	—	—	—	—	—	—	—
	77·7	78·6	69·6	111·7	—	—	46·0	46·0	46·9	—	80·0
	70·6	71·5	67·5	104·5	—	—	48·5	48·1	48·1	—	81·4
	71·9	72·3	74·6	96·4	—	—	51·9	50·9	51·9	—	—
	72·9	73·1	69·8	104·6	80·6	60·0?	47·2	47·2	47·6	78·3	73·0
	75·1	75·1	75·4	99·6	—	—	45·2	46·1	46·5	81·7	79·8
	80·4	81·1	75·0	107·2	78·9	54·7?	42·3	42·7	42·7	—	79·1
	78·6	79·8	67·6	116·3	74·2	51·5	50·5	50·0	52·0	—	78·0
	71·6	73·3	72·2	99·3	85·9	60·8	35·1 ?	35·1 ?	35·7 ?	—	—
	75·9	77·5	71·6	105·9	65·2	44·2	56·5	56·5	55·3	—	69·0
	75·7	76·9	75·1	100·7	66·7	—	44·9	44·9	45·8	89·6	84·1
	78·9	78·9	71·1	110·9	72·8	53·4	45·4	44·9	44·4	—	79·3
	80·1	79·6	70·5	113·6	—	—	—	—	—	—	—
	76·5	76·8	70·6	108·4	59·2	44·6?	59·1	59·1	60·5	—	83·7
	72·4	73·0	73·8	98·2	70·7	49·6	50·0	52·4	51·2	—	89·7
	73·9	75·3	69·3	106·7	—	—	48·5	47·2	48·1	89·2	78·6
	74·5	74·9	72·1	103·3	—	—	—	—	—	—	—
	74·7	75·9	77·1	96·9	65·1	47·9	49·0	48·1	49·0	79·8	74·4
	77·8	78·2	77·8	100·0	71·9	53·9	51·1	51·1	52·2	—	69·2
	71·4	72·9	71·4	100·0	71·8	52·6	49·0	47·2	48·5	—	73·9
	77·3	77·9	68·6	112·7	66·7	48·9	52·1	53·2	52·1	82·1	76·2
	75·9	77·1	68·6	110·7	73·6	53·0	42·9	43·2	42·9	—	84·8
	77·8	79·0	75·1	103·5	69·4	50·6	49·0	49·0	49·0	—	73·9
	74·3	74·6	—	—	—	—	—	—	—	—	—
	78·9	81·4	75·3	104·8	77·0	53·8	48·6	49·1	49·1	—	81·0

	100 B/L	100 B/F	100 H'/L	100 B/H'	100 G'H/GB	100 G'H/J	100 NB/NH'	100 NB/NH, R	100 NB/NH, L	100 O_2/O_1', R	100 O_2/O_1, R
	73·2	73·6	—	—	—	—	—	—	—	—	—
	76·5	76·1	—	—	—	—	—	—	—	—	—
	77·2	78·1	—	—	—	—	—	—	—	—	—
	69·7	69·5	71·5	97·4	—	—	42·6	41·8	42·6	—	—
	72·1	—	—	—	—	—	—	—	—	—	—
	72·3	72·7	70·1	103·1	75·6	51·6	44·2	44·7	44·2	—	74·4
	72·5	73·3	64·9	111·8	—	—	—	—	—	—	—
	—	—	—	—	76·6	—	48·4	48·9	48·4	—	—
	82·1	81·6	74·6	110·1	73·5	—	48·9	48·9	49·4	—	81·3
	—	—	—	—	—	—	—	—	—	—	—
	—	—	—	—	—	—	—	—	—	—	—
	78·6	79·2	68·7	114·4	67·0	47·7	49·4	47·8	48·9	—	78·8
	75·7	77·0	68·7	110·2	63·2	46·9	55·2	54·5	55·2	—	80·5
	83·8	83·8	76·9	109·0	79·5	—	46·0	46·0	46·0	—	—
	77·1	77·5	—	—	—	—	—	—	—	—	—
	73·8	74·0	66·5	111·0	73·2	56·2	47·0	47·0	46·1	93·2	85·0
	75·6	76·6	66·1	114·3	77·6	54·5	56·3	55·1	54·5	89·5	86·0
	70·5	70·5	—	—	84·3	—	—	—	—	—	83·3
	83·2	83·2	74·3	112·0	77·4	—	43·8	44·2	44·7	—	—
	72·2	73·0	71·4	101·2	—	—	—	—	—	—	—
	75·3	75·7	—	—	—	—	—	—	—	—	—
	—	—	—	—	—	—	—	—	—	—	—
	78·2	77·8	73·7	106·2	75·2	—	52·4	52·4	52·4	—	81·0

100 B/L	100 B/F	100 H'/L	100 B/H'	100 $G'H/GB$	100 $G'H/J$	100 NB/NH'	100 $NB/NH, R$	100 $NB/NH, L$	100 $O_2/O_1', R$	100 $O_2/O_1, R$	100 $O_2/O_1, L$	100 G_2/G_1	100 fmb/fml	*Oc. I*	$\angle P$	$\angle N$	$\angle A$	$\angle B$	$\angle\theta_1$	$\angle\theta_2$
73·8	73·8	73·5	100·4	74·9	—	49·1	49·1	48·1	—	—	80·5	—	—	—	84·0	62·0	78·0	40·0	34·0	6·0
72·5	73·3	—	—	—	—	—	—	—	—	—	—	—	71·8	57·3	—	—	—	—	—	—
75·1	76·4	—	—	—	—	—	—	—	—	—	—	—	—	61·8	—	—	—	—	—	—
75·6	76·4	70·2	107·6	—	—	—	—	—	—	—	76·2	—	75·6	57·8	—	—	—	—	—	—
68·8	71·0	—	—	—	—	—	—	—	—	—	—	—	—	59·0	—	—	—	—	—	—
70·0	70·0	66·3	105·5	—	—	—	—	—	—	—	80·5	—	79·5	56·7	—	—	—	—	—	—
69·3	71·6	68·8	100·8	74·2	52·0	50·0	48·0	48·0	—	78·0	78·0	—	—	57·6	83·5	66·5	73·5	40·0	30·0	10·0
68·7	69·4	70·2	97·8	—	—	—	—	—	—	—	—	—	—	57·7	—	—	—	—	—	—
74·4	74·7	69·2	107·4	68·3	50·4	50·9	49·1	48·2	82·5	75·0	77·3	—	64·6	65·7	84·0	67·0	72·0	41·0	29·0	12·0
70·2	73·0	66·2	106·2	77·7	54·1	46·5	46·0	46·5	71·4	69·8	72·1	89·1	85·7	54·9	93·0	58·0	79·0	43·0	29·0	14·0
72·9	—	72·7	100·4	—	—	50·0	50·0	50·6	—	90·0	—	—	79·5	55·6	83·0	66·0	74·5	39·5	33·0	8·5
82·4	82·1	72·2	114·1	68·3	—	47·7	47·7	47·2	83·1	79·0	80·2	—	91·4	54·9	90·0	64·0	75·0	41·0	26·0	15·0
73·6	75·1	73·1	100·7	71·6	51·8	44·7	46·0	44·7	70·6	69·8	69·8	—	84·6	62·1	87·5	67·0	70·0	43·0	25·5	17·5
78·7	78·9	80·1	98·3	72·7	—	39·5	42·1	40·9	—	—	—	—	93·6	64·7	82·0	69·5	70·5	40·0	28·5	11·5
74·7	77·3	74·2	100·7	79·3	—	52·1	51·1	53·3	—	82·5	79·1	—	85·7	59·9	85·0	62·0	78·5	39·5	33·0	6·5
74·9	75·1	69·0	108·6	72·3	—	—	—	—	—	—	—	—	78·0	61·1	90·0	65·0	72·0	43·0	25·0	18·0
76·0	75·7	75·5	100·7	85·7	—	—	—	—	—	—	—	—	87·1	56·5	86·0	57·5	77·5	45·0	36·5	8·5
75·3	76·1	—	—	68·6	—	47·9	49·5	49·5	—	76·9	76·9	—	78·0	54·5	89·0	67·5	77·5	35·0	23·5	11·5
74·1	76·3	74·1	100·0	69·1	—	51·0	53·2	50·5	—	—	73·2	—	—	—	83·0	70·0	75·0	35·0	27·0	8·0
74·4	76·1	70·2	106·0	70·2	49·2	51·9	51·9	51·9	91·9	81·0	80·5	—	86·1	56·9	90·0	61·0	78·0	41·0	29·0	12·0
74·0	75·8	69·7	106·2	71·1	51·9	47·5	47·1	46·2	89·2	82·5	82·5	80·9	79·5	60·5	87·5	61·5	76·5	42·0	31·0	11·0
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
77·7	78·6	69·6	111·7	—	—	46·0	46·0	46·9	—	80·0	81·0	—	78·8	59·9	—	—	—	—	—	—
70·6	71·5	67·5	104·5	—	—	48·5	48·1	48·1	—	81·4	81·8	—	64·6	57·5	—	—	—	—	—	—
71·9	72·3	74·6	96·4	—	—	51·9	50·9	51·9	—	—	—	—	91·2	63·6	82·0	64·0	75·0	41·0	34·0	7·0
72·9	73·1	69·8	104·6	80·6	60·0?	47·2	47·2	47·6	78·3	73·0	77·0	74·1	77·0	56·2	84·0	63·5	71·5	45·0	32·5	12·5
75·1	75·1	75·4	99·6	—	—	45·2	46·1	46·5	81·7	79·8	77·4	—	93·9	57·8	90·0	—	—	—	—	—
80·4	81·1	75·0	107·2	78·9	54·7?	42·3	42·7	42·7	—	79·1	77·4	80·4	90·0	59·7	90·0	63·0	73·5	43·5	27·0	16·5
78·6	79·8	67·6	116·3	74·2	51·5	50·5	50·0	52·0	—	78·0	79·0	—	98·5	55·6	88·0	61·0	76·0	43·0	31·0	12·0
71·6	73·3	72·2	99·3	85·9	60·8	35·1 ?	35·1 ?	35·7 ?	—	—	81·4	—	75·0	58·2	89·0	60·0	75·0	45·0	31·0	14·0
75·9	77·5	71·6	105·9	65·2	44·2	56·5	56·5	55·3	—	69·0	—	—	79·7	60·0	92·0	62·0	82·0	36·0	26·0	10·0
75·7	76·9	75·1	100·7	66·7	—	44·9	44·9	45·8	89·6	84·1	86·3	88·2	82·9	60·2	91·0	59·0	81·0	40·0	30·0	10·0
78·9	78·9	71·1	110·9	72·8	53·4	45·4	44·9	44·4	—	79·3	81·5	83·3	93·9	65·4	90·0	63·5	72·0	44·5	26·5	18·0
80·1	79·6	70·5	113·6	—	—	—	—	—	—	—	—	—	86·5	57·3	—	—	—	—	—	—
76·5	76·8	70·6	108·4	59·2	44·6?	59·1	59·1	60·5	—	83·7	85·7	—	—	58·8	90·0	—	—	—	—	—
72·4	73·0	73·8	98·2	70·7	49·6	50·0	52·4	51·2	—	89·7	82·9	—	81·9	59·6	86·5	64·0	78·5	37·5	29·5	8·0
73·9	75·3	69·3	106·7	—	—	48·5	47·2	48·1	89·2	78·6	78·6	83·7	91·4	56·8	89·0	61·0	75·0	44·0	30·0	14·0
74·5	74·9	72·1	103·3	—	—	—	—	—	—	—	—	—	83·1	62·5	—	—	—	—	—	—
74·7	75·9	77·1	96·9	65·1	47·9	49·0	48·1	49·0	79·8	74·4	73·3	91·5	87·1	60·6	90·0	55·0	85·0	40·0	35·0	5·0
77·8	78·2	77·8	100·0	71·9	53·9	51·1	51·1	52·2	—	69·2	73·7	79·2	90·0	55·1	81·5	67·0	74·0	39·0	31·5	7·5
71·4	72·9	71·4	100·0	71·8	52·6	49·0	47·2	48·5	—	73·9	74·4	—	87·5	61·0	89·0	60·0	78·0	42·0	31·0	11·0
77·3	77·9	68·6	112·7	66·7	48·9	52·1	53·2	52·1	82·1	76·2	75·0	75·0	83·3	57·7	84·0	67·0	75·0	38·0	29·0	9·0
75·9	77·1	68·6	110·7	73·6	53·0	42·9	43·2	42·9	—	84·8	84·8	—	86·8	57·8	85·0	64·0	74·0	42·0	31·0	11·0
77·8	79·0	75·1	103·5	69·4	50·6	49·0	49·0	49·0	—	73·9	78·6	—	89·5	59·2	84·0	63·5	78·5	38·0	32·5	5·5
74·3	74·6	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
78·9	81·4	75·3	104·8	77·0	53·8	48·6	49·1	49·1	—	81·0	77·4	—	87·1	59·9	88·0	63·5	74·0	42·5	28·5	14·0

100 B/L	100 B/F	100 H'/L	100 B/H'	100 $G'H/GB$	100 $G'H/J$	100 NB/NH'	100 $NB/NH, R$	100 $NB/NH, L$	100 $O_2/O_1', R$	100 $O_2/O_1, R$	100 $O_2/O_1, L$	100 G_2/G_1	100 fmb/fml	*Oc. I*	$\angle P$	$\angle N$	$\angle A$	$\angle B$	$\angle\theta_1$	$\angle\theta_2$
73·2	73·6	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
76·5	76·1	—	—	—	—	—	—	—	—	—	—	—	—	61·2	—	—	—	—	—	—
77·2	78·1	—	—	—	—	—	—	—	—	—	—	—	—	58·8	—	—	—	—	—	—
69·7	69·5	71·5	97·4	—	—	42·6	41·8	42·6	—	—	—	—	78·1	56·2	87·0	—	—	—	—	—
72·1	—	—	—	—	—	—	—	—	—	—	—	—	—	61·0	—	—	—	—	—	—
72·3	72·7	70·1	103·1	75·6	51·6	44·2	44·7	44·2	—	74·4	78·0	—	83·8	58·1	88·5	63·5	77·0	39·5	28·0	11·5
72·5	73·3	64·9	111·8	—	—	—	—	—	—	—	—	—	74·4	57·1	—	—	—	—	—	—
—	—	—	—	76·6	—	48·4	48·9	48·4	—	—	89·9	—	—	59·2	—	—	—	—	—	—
82·1	81·6	74·6	110·1	73·5	—	48·9	48·9	49·4	—	81·3	77·5	—	90·9	59·0	79·0	71·0	66·0	43·0	30·0	13·0
—	—	—	—	—	—	—	—	—	—	—	—	—	—	66·4	—	—	—	—	—	—
—	—	—	—	—	—	—	—	—	—	—	—	—	88·2	—	—	—	—	—	—	—
78·6	79·2	68·7	114·4	67·0	47·7	49·4	47·8	48·9	—	78·8	79·5	79·0	82·9	59·2	85·0	70·0	74·0	36·0	25·0	11·0
75·7	77·0	68·7	110·2	63·2	46·9	55·2	54·5	55·2	—	80·5	78·0	—	—	61·4	86·0	—	—	—	—	—
83·8	83·8	76·9	109·0	79·5	—	46·0	46·0	46·0	—	—	—	—	71·6	59·2	92·0	—	—	—	—	—
77·1	77·5	—	—	—	—	—	—	—	—	—	—	—	—	59·2	—	—	—	—	—	—
73·8	74·0	66·5	111·0	73·2	56·2	47·0	47·0	46·1	93·2	85·0	85·0	—	84·8	56·8	87·0	65·0	70·0	45·0	28·0	17·0
75·6	76·6	66·1	114·3	77·6	54·5	56·3	55·1	54·5	89·5	86·0	87·2	—	87·1	57·1	87·0	64·0	72·0	44·0	29·0	15·0
70·5	70·5	—	—	84·3	—	—	—	—	—	83·3	—	—	—	58·6	90·0	—	—	—	—	—
83·2	83·2	74·3	112·0	77·4	—	43·8	44·2	44·7	—	—	—	90·2	77·8	—	90·0	61·0	78·0	41·0	29·0	12·0
72·2	73·0	71·4	101·2	—	—	—	—	—	—	—	—	—	—	63·9	84·0	66·0	73·0	41·0	30·0	11·0
75·3	75·7	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
—	—	—	—	—	—	—	—	—	—	—	—	81·1	—	—	—	—	—	—	—	—
78·2	77·8	73·7	106·2	75·2	—	52·4	52·4	52·4	—	81·0	—	—	87·5	59·4	82·0	66·5	74·5	39·0	31·5	7·5

To face p. 407.

The apparent identity of type that exists between the Dunstable mandibles, on the one hand, and the Anglo-Saxon mandibles from Burwell and from the London Museums, on the other, is of special interest as the corresponding crania when examined were found to be distinctly divergent in type.

We have provided records of a considerable amount of unpublished Anglo-Saxon cranial material and have ventured to draw some inferences from the data we have studied as to the probable relationships of the cranial samples from Bidford-on-Avon, Burwell and the London Museums, but we fully recognise, and are ready to acquiesce in the view, that a final expression of opinion on real diversity or identity of Anglo-Saxon groups from different localities will only be possible when more material is available for study and when all the evidence obtainable from this material, including that provided by cranial contours of the several normae, is utilised.

APPENDIX VII. INDIVIDUAL MEASUREMENTS OF MALE ANGLO-SAXON MANDIBLES FROM BURWELL.

Serial Letter or No.	w_1	g_0g_0	c_rc_r	zz	c_yl	ml	c_pl	rb'	m_2p_1	h_1	m_2h	c_rh	rl	$\angle M$	$\angle R$	$\angle C'$	$100\ c_rh/ml$	$100\ c_rc_r/ml$	$100\ g_0g_0/c_pl$	$100\ g_0g_0/c_rc_r$	$100\ rb'/rl$
$X2$	125·0	97·5	102·5	47·5	21·0	110·0	73·5	28·0	30·5	32·0	24·5	56·0	58·0*	133·0	64·0	52·5	50·9	93·2	132·7	95·1	48·3
$X8$	127·0	102·0	92·5	45·5	22·0*	110·0	80·0	34·5	27·5	30·0	26·0	62·0	65·0	121·0	60·0	59·5	56·4	84·1	127·5	110·3	53·1
10	126·5	96·5	100·0	46·0	23·5	103·5	86·5	34·5	28·5	32·0	32·5	72·0	76·5	110·0	71·0	67·0	69·6	96·6	111·6	96·5	45·1
15	121·5	83·0	95·5	47·5	21·5	97·0	82·0	36·0	27·5	27·0	24·5	62·5	58·0	110·0	75·5	62·5	64·4	98·5	101·2	86·9	62·1
22	129·0	104·0	98·0	43·5	21·0	113·5	75·0	30·0	29·0	28·0	27·0	70·5	74·0	123·5	69·0	69·5	62·1	86·3	138·7	106·1	40·5
23	—	100·5	100·0	47·5	22·0	115·0	87·0	31·5	26·5	29·5	31·0	—	76·0	116·0	—	59·0	—	87·0	115·5	100·5	41·4
27 i	121·5	102·5	95·0	45·0	25·0	101·0	78·5	35·0	28·5*	30·0	28·5	69·0	70·5	112·0	64·0	62·0	68·3	94·1	130·6	107·9	49·6
28	122·0	91·0	96·0	46·0	—	110·0	80·0	32·5	29·5	34·5	28·0	64·0	63·0	124·0	74·0	64·0	58·2	87·3	113·8	94·8	51·6
29	—	91·5	92·5	42·5	—	91·0	68·5	30·0	31·0*	29·0	26·5	61·0	55·0	122·0	—	73·0	67·0	101·7	133·6	98·9	54·5
31	—	95·0	100·5	47·5	18·5	111·0	82·5	35·5	30·0	32·5	27·5	71·5	63·5	114·0	76·5	78·0	65·0	91·4	115·2	94·5	55·9
34	123·5	102·0	100·5	46·0	21·5	107·5	78·5	31·5	28·0	34·5	28·5	65·5	64·0	122·0	74·0	71·5	60·9	93·5	129·9	101·5	49·2
37	120·0	101·5	106·0	47·5	22·0	108·5	76·0	26·0	29·0	28·0	28·0	76·0	68·0	122·0	84·0	57·0	70·0	97·7	133·6	95·8	38·2
41	124·0	101·5	—	44·5	22·5	110·0	82·5	32·0	26·5	30·5	30·0	67·0	70·0	117·0	71·0	61·5	60·9	—	123·0	—	45·7
42	—	98·5	89·5	45·5	20·5*	107·0	77·0	32·0	29·0	35·5	26·5	56·5	68·0	122·0	55·0	67·0	52·8	83·6	127·9	110·1	47·1
43	116·5	94·0	98·0	50·5	17·5	100·5	70·5	33·0	28·5	31·0	27·0	62·0	59·0	125·0	73·0	66·5	61·7	97·5	133·3	95·9	55·9
44	121·0	103·5	—	45·5	20·0	101·0	70·0	27·0*	28·5	30·5	24·5	58·0*	52·0	130·0	73·0	78·0	57·4	—	147·9	—	51·9
47	—	101·5	107·5	49·5	23·5*	107·5	88·5	38·5	30·0	32·5	30·0*	71·5	65·0*	112·0	79·5	71·0	66·5	100·0	114·7	94·4	59·2
56	130·0	106·0	103·5	47·5	22·0	107·0	83·0	36·5	30·5	30·5	33·0	75·0	76·5	112·0	72·0	66·0	70·1	96·7	127·7	102·4	47·7
58	123·0	93·5	106·5	46·0	23·5	98·5	78·0	32·5	28·5	30·5	28·5	69·0	68·0	110·0	73·0	71·5	70·1	108·1	119·9	87·8	47·8
66	123·0	90·5	98·0	45·0	19·5	99·5	78·0	38·0	28·5	33·5	29·0	69·0	63·0	113·0	77·5	73·0	69·3	98·5	116·0	92·3	60·3
67	—	—	101·5	43·5	21·0*	108·5	83·0	30·0	—	36·5	—	61·0	69·0	113·0	66·0	67·0	56·2	93·5	—	—	43·5
68	115·5	95·5	100·5	45·5	19·0*	99·0	76·0	32·5	31·0*	31·0	24·5	66·5	61·0	115·0	80·0	84·0	67·2	101·5	125·7	95·0	53·3
70	125·0	97·5	97·0	48·0	25·5	108·5	74·5	31·5	28·5	35·0	30·5	61·5	62·0	124·0	64·5	77·0	56·7	89·4	130·9	100·5	50·6
74	125·5	91·5	97·5	44·5	20·5*	109·0	79·5	33·5	28·0	34·0	30·5	60·0	56·5	126·0	66·0	70·0	55·0	89·4	115·1	93·8	59·3
77	133·0	115·0	106·5	45·0	24·5	108·5	75·5	34·0	29·0	34·0	27·5	70·0	64·0	125·0	73·5	65·5	64·5	98·2	152·3	108·0	53·1
78	115·5	95·5	100·0	42·5	20·0	92·5	75·0	30·0	27·5	29·0	28·5	68·5	61·5	110·0	88·0	65·0	74·1	108·1	127·3	95·5	48·8
89	120·0	98·0	97·0	44·5	22·5*	107·0	79·5	30·5	27·5*	33·5	25·5	66·0*	68·0*	122·0	67·5	68·5	61·7	90·7	123·3	101·0	44·5
92	120·5	99·0	100·0	43·0	20·0	100·0	77·5	31·5	27·0	30·0	24·5	64·0	59·5	114·0	75·0	59·5	64·0	100·0	127·7	99·0	52·9
93 i	112·0	91·5	96·5	46·5	16·0	95·5	66·5	28·5	29·5*	31·0	24·0	49·0	49·0	132·0	68·0	69·0	51·3	101·0	137·6	94·8	58·2
93 ii	125·0	108·0	97·5	48·5	20·0	106·0	77·0	33·0	30·0	30·5	24·5	55·5	57·5	124·0	62·0	61·5	52·4	92·0	140·3	110·8	57·4
102	120·0	99·0	94·0	44·5	21·5	109·0	77·0	29·5	28·0	30·0	24·5	63·5	63·0	124·0	64·0	67·0	58·3	86·2	128·6	105·3	46·8
104	108·0	90·0	97·0	45·5	19·0	100·5	73·5	30·5	25·5*	—	25·5	75·0	65·5	120·0	83·0	—	74·6	96·5	122·4	92·8	46·6
107	130·5	102·0	107·5	45·5	—	100·0	74·0	28·5	28·0*	32·5	24·0*	57·0	57·0	122·0	77·0	57·0	57·0	107·5	137·8	94·9	50·0
108	123·5	95·5	—	42·5	18·5	102·5	70·0	26·0	—	—	—	—	55·0	127·0	—	—	—	—	136·4	—	47·3
111	—	105·0	—	49·5	21·0	109·0	83·0	35·0	28·5	32·5	28·5	72·5	68·0	113·0	77·5	63·0	66·5	—	126·5	—	51·5
112	123·5	95·5	98·5	45·0	23·0*	109·5	78·0	33·5	28·5	32·0	29·5	66·5	66·0	126·0	73·5	62·0	60·7	90·0	122·4	97·0	50·8
115	125·0	110·5	95·0	46·0	21·5	119·0	72·5	30·0	27·0	35·5	27·0	55·0	63·0	143·0	53·0	64·0	46·2	79·8	152·4	116·3	47·6
121	116·0	92·5	89·5	45·5	20·5	107·5	87·5	37·0	28·5	30·5	26·5	59·0	62·0	112·0	64·0	61·5	54·9	83·3	105·7	103·4	59·7
122	127·0	106·0	92·5	43·5	22·5	105·0	80·5	28·0	27·5	29·0	27·5	72·0	67·0	116·0	76·0	70·5	68·6	88·1	131·7	114·6	41·8
123	123·0	108·5	98·5	49·5	20·5	105·0	81·0	32·5	28·5	30·5	28·5	61·0	60·0	118·0	66·0	65·5	58·1	93·8	134·0	110·2	54·2
128	106·5	92·5	—	42·5	17·5	110·0	79·0	29·5	—	30·5	—	58·0	56·5*	130·0	65·0	72·0	52·7	—	117·1	—	52·2
140	137·0	107·0	107·5	44·5	25·0	113·5	85·5	37·5	27·5	37·0	27·0	72·5	68·5	118·0	74·0	70·0	63·9	94·7	125·1	99·5	54·7

* These measurements were taken on the right side.

APPENDIX VIII. INDIVIDUAL MEASUREMENTS OF FEMALE ANGLO-SAXON MANDIBLES FROM BURWELL.

Serial Letter or No.	w_1	g_0g_0	c_rc_r	zz	c_yl	ml	c_pl	rb'	m_2p_1	h_1	m_2h	c_rh	rl	$\angle M$	$\angle R$	$\angle C'$	$100\ c_rh/ml$	$100\ c_rc_r/ml$	$100\ g_0g_0/c_pl$	$100\ g_0g_0/c_rc_r$	$100\ rb'/rl$
$2a$	—	92·5	107·0	47·5	—	96·5	71·5	30·5	27·5	30·5	28·0*	68·0	—	118·0	—	72·0	70·5	110·9	129·4	86·4	—
$2b$	117·5	95·0	—	44·0	20·5	100·0	71·5	27·5	28·5	27·5	23·0	—	56·0	127·0	—	69·0	—	—	132·9	—	49·1
13	112·0	90·0	86·5	43·5	19·5	108·5	80·5	32·5	26·0	35·5	20·0*	55·5	66·0*	121·0	52·0	75·5	51·2	79·7	111·8	104·0	49·2
14	—	93·5	92·0	43·5	18·0*	92·0	66·5	31·5*	30·0*	27·5	24·5	54·0	57·5*	121·0	66·5	70·0	58·7	100·0	140·6	101·6	54·8
25	121·0	97·5	91·0	47·0	23·0	104·5	78·0	35·5	28·0	31·5	32·0	67·0	71·0	115·0	65·0	69·5	64·1	87·1	125·0	107·1	50·0
26	115·0	89·0	—	40·5	20·5*	100·0	80·0	29·5	28·0	25·5	26·0	65·0*	63·0*	119·0	76·5	62·5	59·6	—	111·3	—	46·8
32	110·0	88·0	91·0	45·5	18·5	94·5	66·5	29·0	28·5	29·5	25·5	58·5	60·5	126·0	67·0	73·5	61·9	96·3	132·3	96·7	47·9
38	117·0	95·5	100·5	44·5	20·0	95·5	65·0	30·0	28·5	26·0	26·5	55·5*	55·0*	130·0	70·0	71·0	58·1	105·2	146·9	95·0	54·5
39	109·0	90·5	85·5	42·5	18·0	100·0	73·0	27·5	29·5	27·0	24·5	55·0	52·0	126·0	68·0	62·0	55·0	85·5	124·0	105·8	52·9
40	120·0	94·0	92·5	42·5	18·0*	102·0	76·5	29·5	27·5*	29·0	24·0*	58·5	54·0	122·0	73·0	70·0	57·4	90·7	122·9	101·6	54·6
52	120·0	90·5	97·0	47·5	21·5	102·0	81·0	31·5	29·5*	25·5	23·5	59·0	61·5	113·0	69·0	62·0	57·8	95·1	111·7	93·3	51·2
53	106·5	81·5	93·5	46·0	19·0	94·0	69·5	32·0	26·5	27·5	25·5	60·0	55·0	123·0	78·0	68·0	63·8	99·5	117·3	87·2	58·2
59	125·0	85·0	—	42·0	22·0*	99·0	73·5	33·0	26·5*	—	28·5	59·5	63·0	117·0	62·0	69·0	60·1	—	115·6	—	52·4
61	108·0	94·5	91·0	44·5	21·0*	101·0	76·0	28·5	—	30·5	—	58·0	54·0*	125·0*	70·0	60·0	57·4	90·1	124·3	103·8	52·6
80	127·0	95·5	100·0	44·5	19·5	102·5	71·5	29·0	27·5	33·5	25·5	56·0	51·0	131·0	71·0	—	54·6	97·6	133·6	95·5	56·9
97	—	92·5	91·5	45·0	19·0	106·5	80·0	33·5	—	—	—	61·0	59·0*	125·0	75·0	60·0	57·3	85·9	115·6	101·1	56·8
119	118·0	88·0	89·5	42·0	17·0	100·0	73·5	31·5	26·5*	28·5	22·0*	58·5	60·0*	120·0	67·0	69·0	58·5	89·5	119·7	98·3	52·5
135	—	—	89·0	41·0	16·5*	99·0	75·0	32·5	30·5	31·5	27·0	—	49·5?	118·0	—	78·0	—	89·9	—	—	65·7
Y Wants) No.	115·0	92·0	—	46·0	18·0	95·0	69·0	28·5	28·5	29·0	24·0	50·0	47·0	129·0	68·0	60·0	52·6	—	133·3	—	60·6
	114·0	98·0	88·5	41·5	15·5	97·0	70·0	29·5	25·5	30·5	25·0	52·5*	53·0	126·0	65·0	68·0	54·1	91·2	140·0	110·7	55·7

* These measurements were taken on the right side.

ON THE NUMERICAL EVALUATION OF HIGH ORDER INCOMPLETE EULERIAN INTEGRALS.

By KARL AND MARGARET V. PEARSON.

I. *The Incomplete* Γ*-Function.*

The Incomplete Γ-Function is defined by

$$\Gamma_{x_0}(p) = \int_0^{x_0} e^{-x} x^{p-1}\, dx \quad \text{.....(i)},$$

and the Incomplete Γ-Function Ratio by

$$I'(x_0, p) = \int_0^{x_0} e^{-x} x^{p-1}\, dx/\Gamma(p) \quad \text{.....(ii)}.$$

The Ratio is the probability integral of the frequency curve

$$y = y_0 e^{-x} x^{p-1} \quad \text{.....(iii)}.$$

This curve has for its characteristics

$$\text{Mode} = \tilde{x} = p-1, \quad \text{Mean} = \bar{x} = p, \quad \text{and Standard Deviation} = \sqrt{p} \text{...(iv)}.$$

In the *Tables of the Incomplete* Γ*-Function** the notation is somewhat different. Therein

$$I(u, p) = \int_0^{u\sqrt{p+1}} e^{-x} x^p\, dx/\Gamma(p+1),$$

or

$$I\left(\frac{x_0}{\sqrt{p}}, p-1\right) = \int_0^{x_0} e^{-x} x^{p-1} dx/\Gamma(p) = I'(x_0, p) \quad \text{.....(v)}.$$

Let us transfer (iii) to its mode and then integrate it up to a distance x' beyond the mode, where actually x' may be positive or negative.

$$\begin{aligned} I'(x_0, p)\,\Gamma(p) &= \int_0^{p-1} e^{-x} x^{p-1} dx + \int_0^{x_0'} e^{-(p-1)-x'} (p-1+x')^{p-1} dx' \\ &= I\left(\frac{p-1}{\sqrt{p}}, p-1\right)\Gamma(p) + e^{-(p-1)}(p-1)^{p-1} \int_0^{x_0'} \left\{e^{-\frac{x'}{p-1}}\left(1+\frac{x'}{p-1}\right)\right\}^{p-1} dx'. \end{aligned}$$

Now let us take

$$x'/(p-1) = t \quad \text{.....(vi)}.$$

Then

$$\begin{aligned} I'(x_0, p) &= I\left(\frac{p-1}{\sqrt{p}}, p-1\right) + \frac{\sqrt{2\pi}\, e^{-(p-1)}(p-1)^{p-\frac{1}{2}}}{\Gamma(p)} \frac{\sqrt{p-1}}{\sqrt{2\pi}} \int_0^{t_0} \{e^{-t}(1+t)\}^{p-1} dt \\ &= I\left(\frac{p-1}{\sqrt{p}}, p-1\right) + \frac{P_0\sqrt{p-1}}{\sqrt{2\pi}} \int_0^{t_0} W_t dt \quad \text{.....(vii)}, \end{aligned}$$

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where $$t_0 = x_0'/(p-1) = \frac{x_0-(p-1)}{p-1} \quad \text{.....(viii),}$$

$$P_0 = \frac{\sqrt{2\pi}\, e^{-(p-1)}\,(p-1)^{p-\frac{1}{2}}}{\Gamma(p)} \quad \text{.....(ix),}$$

and is, for $p = 50$ and upwards, only slightly less than unity;

$$W_t = \{e^{-t}(1+t)\}^{p-1},$$

$$\log W_t = (p-1)\{-t+\log(1+t)\} = (p-1)\left(-\frac{t^2}{2}+\frac{t^3}{3}-\frac{t^4}{4}+\dots\right),$$

or
$$W_t = e^{-\frac{1}{2}(p-1)t^2} \times e^{(p-1)\left(\frac{t^3}{3}-\frac{t^4}{4}+\frac{t^5}{5}-\dots\right)}$$

$$= e^{-\frac{1}{2}(p-1)t^2}\,(1+T+\tfrac{1}{2}T^2+\tfrac{1}{6}T^3+\tfrac{1}{24}T^4+\dots) \quad \text{.....(x),}$$

where $$T = (p-1)\frac{t^3}{3}\,(1-\tfrac{3}{4}t+\tfrac{3}{5}t^2-\dots) \quad \text{.....(xi).}$$

The part of W_t independent of T will give us

$$\frac{P_0\sqrt{p-1}}{\sqrt{2\pi}}\int_0^{t_0} e^{-\frac{1}{2}(p-1)t^2}\,dt.$$

Taking $\zeta^2 = (p-1)t^2$, this becomes

$$P_0\int_0^{\zeta_0}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\zeta^2}\,d\zeta \quad \text{.....(xii),}$$

where $$\zeta_0 = \sqrt{p-1}\;t_0 = \frac{x_0-(p-1)}{\sqrt{p-1}} \quad \text{.....(xiii).}$$

This can be found at once from the Tables of the Normal Probability Integral

$$= P_0\,\tfrac{1}{2}\alpha_{\zeta_0} \quad \text{.....(xiv),}$$

where $\frac{1}{2}\alpha_{\zeta_0}$ is always less than (or equal to) $\frac{1}{2}$.

We now turn to the remainder of W_t. This gives

$$W_T = e^{-\frac{1}{2}(p-1)t^2}\left(T+\frac{1}{2}T^2+\frac{1}{6}T^3+\frac{1}{24}T^4+\dots\right)$$

$$= e^{-\frac{1}{2}(p-1)t^2}\left\{\frac{1}{3}(p-1)t^3\frac{1}{4}(p-1)t^4+\frac{1}{5}(p-1)t^5-\left(\frac{1}{6}-\frac{1}{18}(p-1)\right)(p-1)t^6\right.$$
$$+\left(\frac{1}{7}-\frac{1}{12}(p-1)\right)(p-1)t^7-\left(\frac{1}{8}-\frac{47}{480}(p-1)\right)(p-1)t^8$$
$$+\left(\frac{1}{9}-\frac{19}{180}(p-1)+\frac{1}{162}(p-1)^2\right)(p-1)t^9$$
$$-\left(\frac{1}{10}-\frac{153}{1400}(p-1)+\frac{1}{72}(p-1)^2\right)(p-1)t^{10}$$
$$+\left(\frac{1}{11}-\frac{31}{280}(p-1)+\frac{31}{1440}(p-1)^2\right)(p-1)t^{11}$$
$$\left.-\left(\frac{1}{12}-\frac{3349}{30240}(p-1)+\frac{493}{17280}(p-1)^2-\frac{1}{1944}(p-1)^3\right)(p-1)t^{12}+\dots\right\}$$

.....(xv),

and we need

$$R_T = \frac{e^{-(p-1)}(p-1)^{p-1}}{\Gamma(p)} \int_0^{x_0'} W_T dx' = \frac{e^{-(p-1)}(p-1)^{p}}{\Gamma(p)} \int_0^{t_0} W_T dt \quad \ldots(\text{xvi}),$$

where $t_0 = x_0'/(p-1) = (x_0 - (p-1))/(p-1)$.

We will now change our variable to z, where

$$z = \tfrac{1}{2}(p-1)t^2 \text{ and } z_0 = \frac{\tfrac{1}{2}(p-1)(x_0-(p-1))^2}{(p-1)^2} = \tfrac{1}{2}\frac{x_0'^2}{p-1} \quad \ldots(\text{xvii}),$$

with x_0' representing the distance from the mode to the boundary of our original Γ-function.

Since $t = \left(\frac{2z}{p-1}\right)^{\frac{1}{2}}$, we find $dt = \tfrac{1}{2}\sqrt{\frac{2}{p-1}}\, z^{-\frac{1}{2}} dz$,

and thus we have

$$R_T = \frac{1}{\sqrt{2}} \frac{e^{-(p-1)}(p-1)^{p-\frac{1}{2}}}{\Gamma(p)} \int_0^{z_0} z^{-\frac{1}{2}} W_T dz$$

$$= \frac{1}{\sqrt{2\pi}} P_0 \frac{1}{\sqrt{2}} \int_0^{z_0} z^{-\frac{1}{2}} W_T dz.$$

Substituting z for t in (xv), we find

$$I'(x_0, p) = \int_0^{x_0} e^{-x} x^{p-1} dx/\Gamma(p)$$

$$= I\left(\frac{p-1}{\sqrt{p}}, p-1\right) + P_0(\tfrac{1}{2}a_{\zeta_0}) + a_1 I(\cdot 7071{,}0678\, z_0, 1)$$
$$+ a_{1\cdot5} I(\cdot 6324{,}5553\, z_0, 1\cdot5) + a_2 I(\cdot 5773{,}5027\, z_0, 2)$$
$$+ a_{2\cdot5} I(\cdot 5345{,}2248\, z_0, 2\cdot5) + a_3 I(\cdot 5000{,}0000\, z_0, 3)$$
$$+ a_{3\cdot5} I(\cdot 4714{,}0452\, z_0, 3\cdot5) + a_4 I(\cdot 4472{,}1360\, z_0, 4)$$
$$+ a_{4\cdot5} I(\cdot 4264{,}0143\, z_0, 4\cdot5) + a_5 I(\cdot 4082{,}4828\, z_0, 5)$$
$$+ a_{5\cdot5} I(\cdot 3922{,}3227\, z_0, 5\cdot5) + a_6 I(\cdot 3779{,}6447\, z_0, 6) \ldots(\text{xviii}),$$

where $\zeta_0 = \frac{x_0'}{\sqrt{p-1}}$, $z_0 = \tfrac{1}{2}\frac{x_0'^2}{p-1}$ as in (xvii),

$$P_0 = \sqrt{2\pi(p-1)}\,(p-1)^{p-1} e^{-(p-1)}/\Gamma(p),$$

and equals by Stirling's Theorem

$$\left(1 + \frac{1}{12}\frac{1}{(p-1)} + \frac{1}{288}\frac{1}{(p-1)^2} - \frac{139}{51840}\frac{1}{(p-1)^3} + \ldots\right)^{-1},$$

which will generally be adequate for computing P_0.

Lastly we have

$$a_1 = \cdot 26596{,}15203\, \frac{P_0}{\sqrt{p-1}}, \qquad a_{1\cdot5} = -\cdot 37500{,}00000\, \frac{P_0}{(p-1)},$$

$$a_2 = \cdot 63830,76486\ \frac{P_0}{(p-1)^{\frac{1}{2}}}, \qquad a_{2\cdot5} = \cdot 41666,66667\left(1-\frac{3}{p-1}\right)\frac{P_0}{p-1},$$

$$a_3 = -1\cdot 59576,91216\left(1-\frac{12}{7}\,\frac{1}{p-1}\right)\frac{P_0}{(p-1)^{1\cdot5}},$$

$$a_{3\cdot5} = 5\cdot 140625\left(1-\frac{60}{47}\,\frac{1}{p-1}\right)\frac{P_0}{(p-1)^2},$$

$$a_4 = \cdot 9456,40961\left(1-\frac{171}{10}\,\frac{1}{p-1}+18\,\frac{1}{(p-1)^2}\right)\frac{P_0}{(p-1)^{1\cdot5}},$$

$$a_{4\cdot5} = -6\cdot 5625\left(1-\frac{1377}{175}\,\frac{1}{p-1}+\frac{36}{5}\,\frac{1}{(p-1)^2}\right)\frac{P_0}{(p-1)^2},$$

$$a_5 = 32\cdot 9792,2852\left(1-\frac{36}{7}\,\frac{1}{(p-1)}+\frac{1440}{341}\,\frac{1}{(p-1)^2}\right)\frac{P_0}{(p-1)^{2\cdot5}},$$

$$a_{5\cdot5} = 2\cdot 67361,11111\left(1-\frac{4437}{80\,(p-1)}+\frac{30141}{140\,(p-1)^2}-\frac{162}{(p-1)^3}\right)\frac{P_0}{(p-1)^2},$$

$$a_6 = -28\cdot 3692,2883\left(1-\frac{15759}{700\,(p-1)}+\frac{2493}{35\,(p-1)^2}-\frac{648}{13\,(p-1)^3}\right)\frac{P_0}{(p-1)^{2\cdot5}}$$

.....(xix).

Illustration I. Let us now test the above results on a sample case; we will take the worst possible case, namely $p = 50$ on the border of the Γ-Function Tables, and find from those tables the value of the required Γ-function ratio by which to test our expansion. Let us take x_0' considerable, $= 2\sqrt{p-1}$, i.e. $= 14$. Accordingly

$$\zeta_0 = \frac{x_0'}{\sqrt{p-1}} = 2, \text{ and } z_0 = \tfrac{1}{2}\frac{x_0'^2}{p-1} = 2,$$

also. Further, $x_0 = (p-1) + x_0' = 63$, and the actual value

$$= I'(63, 50) = I'\left(\frac{63}{\sqrt{p}}, p-1\right) = I\,(\cdot 8909,54543, 49) = \cdot 959,5825$$

by interpolation from the *Incomplete* Γ-*Function Tables.*

Again, $I\left(\dfrac{p-1}{\sqrt{p}}, p-1\right)$, the area up to the mode,

$$= I\left(\frac{49}{\sqrt{50}}, 49\right) = I\,(6\cdot 9296,4646, 49),$$

and from the same tables, $= \cdot 462,1044.$

Calculating P_0, we find for its value, whether from the full value or Stirling's approximation, ·9983,00788. The normal term is

$$P_0 \times \int_0^{\zeta_0} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}\zeta^2}\, d\zeta = P_0 \times \cdot 477,2499,$$

since $\zeta_0 = 2$. Thus this term = ·476,4390. Calculating the a_s's and $I'\left(\frac{z_0}{\sqrt{s+1}}, s\right)$'s, we find

	a_s	I_s	$a_s \times I_s$
a_1	+·0379,2994	$I(1{\cdot}4142,1356, 1)$ = ·593,9947	+·022,5302
$a_{1\cdot5}$	−·0076,4006	$I(1{\cdot}2649,1106, 1{\cdot}5)$ = ·450,5816	−·003,4425
a_2	+·0018,5779	$I(1{\cdot}1547,0054, 2)$ = ·323,3226	+·000,6007
$a_{2\cdot5}$	+·0079,6922	$I(1{\cdot}0690,4497, 2{\cdot}5)$ = ·220,2225	+·001,7550
a_3	−·0044,8199	$I(1{\cdot}0000,0000, 3)$ = ·142,8765	−·000,6404
$a_{3\cdot5}$	+·0020,8171	$I(0{\cdot}9428,0909, 3{\cdot}5)$ = ·088,5874	+·000,1844
a_4	+·0018,1243	$I(0{\cdot}8944,2719, 4)$ = ·052,6530	+·000,0954
$a_{4\cdot5}$	−·0022,9861	$I(0{\cdot}8528,0287, 4{\cdot}5)$ = ·030,0831	−·000,0691
a_5	+·0017,5674	$I(0{\cdot}8164,9658, 5)$ = ·016,5637	+·000,0291
$a_{5\cdot5}$	−·0000,4846	$I(0{\cdot}7844,6454, 5{\cdot}5)$ = ·008,8087	−·000,0004
a_6	−·0009,6015	$I(0{\cdot}7559,2895, 6)$ = ·004,5338	−·000,0045

These include all the terms up to the order $\frac{1}{(p-1)^{2\cdot5}}$.

We will now sum these terms. The sum of the $I\left(\frac{p-1}{\sqrt{p}}, p-1\right)$ and the normal term is ·938,5434. Adding in the a_1 term, we have ·961,0736, whence we proceed as follows:

Terms included	Approximate value	Size and place of error	Terms included	Approximate value	Size and place of error
Up to a_1	·961,0736	+1 in 3rd	Up to a_4	·959,6262	+4 in 5th
„ $a_{1\cdot5}$	·957,6311	−2 in 3rd	„ $a_{4\cdot5}$	·959,5571	−2 in 5th
„ a_2	·958,2318	−1 in 3rd	„ a_5	·959,5862	+4 in 6th
„ $a_{2\cdot5}$	·959,9868	+4 in 4th	„ $a_{5\cdot5}$	·959,5858	+3 in 6th
„ a_3	·959,3464	−2 in 4th	„ a_6	·959,5813	−1 in 6th
„ $a_{3\cdot5}$	·959,5308	−5 in 5th	True value	·959,5825	0 in 7th

Thus with 13 terms of the expansion (xviii), we have only an error of a unit in the sixth decimal place. Accordingly if a table* could be computed of P_0 and the $I\left(\frac{p-1}{\sqrt{p}}, p-1\right)$, together with the first eleven a's, all values of the Incomplete Γ-function could be thrown back on the existing tables for values of $p > 50$, with an accuracy of the order ·000,001, or thereabouts.

Of course one fully admits that this expansion of high order Γ-functions in terms of low order Γ-functions suffers from the same defects as expansions in incomplete normal moment functions, or in tetrachoric functions. But such a table as we have spoken of would probably provide all the accuracy needful in the majority of statistical problems, and would be less laborious than the use of quadrature formulae.

* It might be needful to form the table for every unit value of p from 50 to 100, but later it would probably be adequate to take the intervals at several units distance.

The table above indicates that with every additional a_s term included the approximation becomes closer. Further it indicates that for examples such as our first illustration, we shall have accuracy to the third decimal place if we include terms up to $a_{2\cdot5}$, to the fourth decimal place if we proceed to $a_{3\cdot5}$, while up to a_5 we have accuracy to the fifth decimal place.

Illustration II. We must work another example to illustrate the process, namely $p = 101$, and we will take our $x_0 = 120$, i.e. $= p - 1 + 2\sqrt{p-1}$, so that we are finding

$$\int_0^{120} e^{-x} x^{100}\, dx/\Gamma(101).$$

By choosing $x_0' = 20 = 2\sqrt{p-1}$, our entire series of $I\left(\frac{z_0}{\sqrt{s+1}}, s\right)$ remain the same and need not be recalculated, all the a_s's, however, are changed as well as P_0.

We find $P_0 = \cdot9991,6702$, and since $\frac{1}{2}a_{\zeta_0}$ remains the same, the normal term contribution is

$$\cdot9991,6702 \times \cdot477,2499 = \cdot476,8524.$$

Further* $$\int_0^{p-1} e^{-x} x^{p-1}\, dx/\Gamma(p) = \cdot473,4402.$$

Together, these give us ·950,2926.

We will now form a table as before:

	Value of a_s	Value of I_s	$a_s I_s$	Value up to a_s	Exact value less series value up to a_s
a_1	+·0265,73998	·593,9947	+·015,7848	·966,0774	−·000,7453
$a_{1\cdot5}$	−·0037,46876	·450,5816	−·001,6883	·964,3891	+·000,9430
a_2	+·0006,37776	·323,3226	+·000,2062	·964,5953	+·000,7368
$a_{2\cdot5}$	+·0040,38300	·220,2225	+·000,8893	·965,4846	−·000,1525
a_3	−·0015,67107	·142,8765	−·000,2239	·965,2607	+·000,0714
$a_{3\cdot5}$	+·0005,07077	·088,5874	+·000,0449	·965,3056	+·000,0265
a_4	+·0007,84984	·052,6530	+·000,0412	·965,3468	−·000,0147
$a_{4\cdot5}$	−·0006,04581	·030,0831	−·000,0182	·965,3286	+·000,0035
a_5	+·0003,12710	·016,5637	+·000,0052	·965,3338	−·000,0017
$a_{5\cdot5}$	−·0001,19509	·008,8087	−·000,0011	·965,3327	−·000,0006
a_6	−·0002,21647	·004,5338	−·000,0010	·965,3317	+·000,0004

The exact value was found by quadratures from $x_0 = 120$ to the end of the curve to be ·034,6679, and $1 - \cdot034,6679 = \cdot965,3321$. It will be seen that our final result, using up to the a_6 term, is only four units out in the seventh decimal place. Up to $a_{3\cdot5}$ we are only out two units in the fifth decimal place, and up to $a_{2\cdot5}$ only between one and two units in the fourth decimal place.

It will be clear to the reader that to replace an incomplete Γ-function by a normal curve will not give a correct result even in the *second* decimal place, when $p = 101$, if we want an area reaching to about twice the standard deviation from

* Found by quadrature from 45 ordinates at unit distance apart.

the mode. It would work better, of course, if we knew the area up to the mode, and were only seeking areas up to $\pm\sqrt{p}$ on either side of it. But clearly the safest way to reach high order incomplete Γ-functions is to form a table of $\int_0^{p-1} \frac{e^{-x}x^{p-1}dx}{\Gamma(p)}$ and of the functions of p, i.e. P_0, a_1, $a_{1\cdot5}$ ... a_6, and so throw such incomplete Γ-functions back on the already published table.

Will any trained computer volunteer for the work?

II. *The Incomplete* B-*Function.*

Let the required integral be the B-function ratio

$$I_{x_0}(p+1, q+1) = \int_0^{x_0} \frac{x^p(1-x)^q\,dx}{\mathrm{B}(p+1, q+1)} \quad \text{.....(i),}$$

where $\mathrm{B}(p+1, q+1) = \Gamma(p+1)\Gamma(q+1)/\Gamma(p+q+2)$.

We will suppose $q > p$, which it is always feasible to arrange, if needful, by aid of the transformation $x = 1 - x'$, and finding the area from the other end of the curve.

We have

$$\begin{aligned} I_x(p+1, q+1) &= \frac{\Gamma(p+q+2)}{\Gamma(p+1)\Gamma(q+1)} \int_0^{x_0} x^p e^{q\log(1-x)}\,dx \\ &= \frac{\Gamma(p+q+2)}{\Gamma(p+1)\Gamma(q+1)} \int_0^{x_0} x^p e^{-q\left(x+\frac{1}{2}x^2+\frac{1}{3}x^3+\ldots\right)}\,dx \\ &= \frac{\Gamma(p+q+2)}{\Gamma(p+1)\Gamma(q+1)} \int_0^{x_0} x^p e^{-qx} e^{-q\frac{x^2}{2}-q\frac{x^3}{3}-q\frac{x^4}{4}-\cdots}\,dx. \end{aligned}$$

We may now expand the exponential

$$e^{-q\frac{x^2}{2}-q\frac{x^3}{3}-q\frac{x^4}{4}-\ldots}$$

as equal to $1 + c_2x^2 + c_3x^3 + \ldots + c_sx^s$ ....(ii),

and our integral becomes

$$\begin{aligned} I_{x_0}(p+1, q+1) &= \frac{\Gamma(p+q+2)}{\Gamma(p+1)\Gamma(q+1)} \int_0^{x_0} x^p e^{-qx}(1 + c_2x^2 + c_3x^3 + \ldots + c_sx^s + \ldots)\,dx \\ &= \frac{\Gamma(p+q+2)}{\Gamma(q+1)} \left\{ \int_0^{x_0} \frac{x^p e^{-qx}\,dx}{\Gamma(p+1)} + c_2 \int_0^{x_0} \frac{x^{p+2}e^{-qx}\,dx}{\Gamma(p+1)} + \ldots \right. \\ &\qquad \left. + c_s \int_0^{x_0} \frac{x^{p+s}e^{-qx}\,dx}{\Gamma(p+1)} + \ldots \right\} \\ &= \frac{\Gamma(p+q+2)}{\Gamma(q+1)} \left\{ \frac{I'(z_0, p+1)}{q^{p+1}} + c_2 \frac{I'(z_0, p+3)}{q^{p+3}}(p+2)(p+1) \right. \\ &\qquad + c_3 \frac{I'(z_0, p+4)}{q^{p+4}}(p+3)(p+2)(p+1)\ldots \\ &\qquad \left. + c_s \frac{I'(z_0, p+s+1)}{q^{p+s+1}}(p+s)(p+s-1)\ldots(p+1) + \ldots \right\} \\ &\qquad\qquad \text{.....(iii),} \end{aligned}$$

where we have taken $z = qx$ and $z_0 = qx_0$.

If we now use the notation of the *Tables of the Incomplete* Γ-*Function* we have, on putting $p-1$ for p, and $q-1$ for q,

$$I_{x_0}(p,q) = \int_0^{x_0} x^{p-1}(1-x)^{q-1}\,dx/\mathrm{B}(p,q)$$

$$= \frac{\Gamma(p+q)}{\Gamma(q)}\left\{\frac{I\left(\frac{x_0(q-1)}{\sqrt{p}}, p-1\right)}{(q-1)^p} + c_2 I\left(\frac{x_0(q-1)}{\sqrt{p+2}}, p+1\right)\frac{(p+1)p}{(q-1)^{p+2}}\right.$$

$$+ c_3 I\left(\frac{x_0(q-1)}{\sqrt{p+3}}, p+2\right)\frac{(p+2)(p+1)p}{(q-1)^{p+3}} + \ldots$$

$$\left. + c_s I\left(\frac{x_0(q-1)}{\sqrt{p+3}}, p+s-1\right)\frac{(p+s-1)\ldots(p+1)}{(q-1)^{p+s}} + \ldots\right\} \quad \ldots\ldots\text{(iv)}$$

Thus if $p<50$ and $q>50$, we have reduced our integral to a series of incomplete Γ-function ratios, which can be taken out of the above-mentioned tables. We have to determine the c's and to consider the convergency of the series.

Turning back to (ii) and taking logarithmic differentials, we have

$$(-qx-qx^2-qx^3-\ldots-qx^s-\ldots)(1+c_2x^2+c_3x^3+\ldots+c_sx^s+\ldots)$$
$$= 2c_2x+3c_3x^2+\ldots+3c_sx^{s-1}+\ldots.$$

Hence we easily find $c_2 = -\frac{1}{2}q$, $c_3 = -\frac{1}{3}q$, and generally

$$c_s = \frac{s-1}{s}\,c_{s-1} - \frac{q}{s}\,c_{s-2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(v)}.$$

To suit the notation of (iv), we must, however, put $q-1$ for q, and we then have

$$c_2 = -\frac{q-1}{2}, \quad c_3 = -\frac{q-1}{3}, \quad c_4 = \frac{1}{8}(q-1)^2\left(1-\frac{2}{q-1}\right),$$

$$c_5 = \frac{1}{6}(q-1)^2\left(1-\frac{6}{5(q-1)}\right), \quad c_6 = -\frac{1}{48}(q-1)^3\left(1-\frac{26}{3(q-1)}+\frac{8}{(q-1)^2}\right),$$

$$c_7 = -\frac{1}{24}(q-1)^3\left(1-\frac{22}{5(q-1)}+\frac{24}{7(q-1)^2}\right),$$

$$c_8 = \frac{1}{384}(q-1)^4\left(1-\frac{68}{3(q-1)}+\frac{696}{10(q-1)^2}-\frac{48}{(q-1)^3}\right),$$

$$c_9 = \frac{1}{144}(q-1)^4\left(1-\frac{472}{45(q-1)}+\frac{892}{35(q-1)^2}-\frac{16}{(q-1)^3}\right),$$

$$c_{10} = -\frac{1}{3840}(q-1)^5\left(1-\frac{140}{3(q-1)}+\frac{964}{3(q-1)^2}-\frac{23088}{35(q-1)^3}+\frac{384}{(q-1)^4}\right),$$

$$c_{11} = -\frac{1}{1152}(q-1)^5\left(1-\frac{916}{45(q-1)}+\frac{3716}{35(q-1)^2}-\frac{6704}{35(q-1)^3}+\frac{1152}{11(q-1)^4}\right),$$

$$c_{12} = \frac{1}{46080}(q-1)^6\left(1-\frac{250}{3(q-1)}+\frac{28828}{27(q-1)^2}-\frac{478{,}024}{105(q-1)^3}+\frac{155{,}552}{21(q-1)^4}-\frac{3840}{(q-1)^5}\right)$$
$$\ldots\ldots\text{(vi)}.$$

Now let us write (iv) in the form

$$I_{x_0}(p,q) = a_0 I\left(x_0(q-1)/\sqrt{p}, p-1\right) + a_2 I\left(x_0(q-1)/\sqrt{p+2}, p+1\right) + a_3 I\left(x_0(q-1)/\sqrt{p+3}, p+2\right) + \ldots + a_s I\left(x_0(q-1)/\sqrt{p+s}, p+s-1\right) + \ldots \quad \ldots\ldots\text{(vii)}.$$

Then we have

$$a_0 = \frac{\Gamma(p+q)}{\Gamma(q)}\frac{1}{(q-1)^p}, \quad a_2 = -\frac{\Gamma(p+q)}{\Gamma(q)}\frac{(p+1)p}{2(q-1)^{p+1}},$$

$$a_3 = -\frac{\Gamma(p+q)}{\Gamma(q)}\frac{(p+2)(p+1)p}{3(q-1)^{p+2}},$$

$$a_4 = +\frac{\Gamma(p+q)}{\Gamma(q)}\frac{(p+3)(p+2)(p+1)p}{8(q-1)^{p+2}}\left(1-\frac{2}{q-1}\right),$$

$$a_5 = +\frac{\Gamma(p+q)}{\Gamma(q)}\frac{(p+4)(p+3)(p+2)(p+1)p}{6(q-1)^{p+3}}\left(1-\frac{6}{5(q-1)}\right),$$

$$a_6 = -\frac{\Gamma(p+q)}{\Gamma(q)}\frac{(p+5)(p+4)(p+3)(p+2)(p+1)p}{48(q-1)^{p+3}}\left(1-\frac{26}{3(q-1)}+\frac{8}{(q-1)^2}\right),$$

$$a_7 = -\frac{\Gamma(p+q)}{\Gamma(q)}\frac{(p+6)(p+5)\ldots(p+1)p}{24(q-1)^{p+4}}\left(1-\frac{22}{5(q-1)}+\frac{24}{7(q-1)^2}\right),$$

$$a_8 = +\frac{\Gamma(p+q)}{\Gamma(q)}\frac{(p+7)(p+6)\ldots(p+1)p}{384(q-1)^{p+4}}\left(1-\frac{68}{3(q-1)}+\frac{696}{10(q-1)^2}-\frac{48}{(q-1)^3}\right),$$

$$a_9 = +\frac{\Gamma(p+q)}{\Gamma(q)}\frac{(p+8)(p+7)\ldots(p+1)p}{144(q-1)^{p+5}}\left(1-\frac{472}{45(q-1)}+\frac{892}{35(q-1)^2}-\frac{16}{(q-1)^3}\right),$$

$$a_{10} = -\frac{\Gamma(p+q)}{\Gamma(q)}\frac{(p+9)(p+8)\ldots(p+1)p}{3840(q-1)^{p+5}} \times\left(1-\frac{140}{3(q-1)}+\frac{964}{3(q-1)^2}-\frac{23088}{35(q-1)^3}+\frac{384}{(q-1)^4}\right),$$

$$a_{11} = -\frac{\Gamma(p+q)}{\Gamma(q)}\frac{(p+10)(p+9)\ldots(p+1)p}{1152(q-1)^{p+6}} \times\left(1-\frac{916}{45(q-1)}+\frac{3716}{35(q-1)^2}-\frac{6704}{35(q-1)^3}+\frac{1152}{11(q-1)^4}\right),$$

$$a_{12} = +\frac{\Gamma(p+q)}{\Gamma(q)}\frac{(p+11)(p+10)\ldots(p+1)p}{46080(q-1)^{p+6}} \times\left(1-\frac{250}{3(q-1)}+\frac{28828}{27(q-1)^2}-\frac{478{,}024}{105(q-1)^3}+\frac{155{,}552}{21(q-1)^4}-\frac{3840}{(q-1)^5}\right) \quad \ldots\ldots\text{(viii)}.$$

Illustration (i). As a first example, we will find the incomplete B-function ratio

$$I_{.024}(4, 101) = \int_0^{.024} x^3(1-x)^{100}\,dx/\mathrm{B}(4, 101),$$

or, if $x' = (1-x)$,

$$= \int_{.976}^{1} (1-x')^3 x'^{100}\,dx'/\mathrm{B}(4, 101).$$

This integral is fairly easy to find by the expansion of $(1-x')^3$ and direct integration, using many figure logarithms to determine $(\cdot 976)^{101}$, which equals $\cdot 0859,8675,99$. Thus we obtain

$$I_{\cdot 024}(4, 101) = \cdot 2402,41262.$$

We now proceed to compute our a's from (viii) and determine the I's by interpolation. The results are given in the table below:

	a_s	I_s	Values from Γ-Tables	$a_s \times I_s$	Value of $S(a_s I_s)$ up to a_s
a_0	$+1\cdot 1035,5024$	$I_0(1\cdot 2, 3)$	$=\cdot 2212,7710$	$\cdot 2441,90397$	$\cdot 2441,90397$
a_2	$-\cdot 1103,55024$	$I_2(\cdot 9797,95896, 5)$	$=\cdot 0356,7257$	$-\cdot 0039,36647$	$\cdot 2402,53750$
a_3	$-\cdot 0044,14201$	$I_3(\cdot 9071,14735, 6)$	$=\cdot 0115,9372$	$-\cdot 0000,51180$	$\cdot 2402,02570$
a_4	$+\cdot 0113,55532$	$I_4(\cdot 8485,28136, 7)$	$=\cdot 0033,3734$	$+\cdot 0000,37897$	$\cdot 2402,40467$
a_5	$+\cdot 0012,21145$	$I_5(\cdot 8, 8)$	$=\cdot 0008,6210$	$+\cdot 0000,01053$	$\cdot 2402,41520$
a_6	$-\cdot 0012,71078$	$I_6(\cdot 7589,46648, 9)$	$=\cdot 0002,0019$	$-\cdot 0000,00254$	$\cdot 2402,41266$
a_7	$-\cdot 0002,65953$	$I_7(\cdot 7236,27229, 10)$	$=\cdot 0000,4249$	$+\cdot 0000,00011$	$\cdot 2402,41277$
				True value	$=\cdot 2402,41262$

Now since our *Table of the Incomplete* Γ*-Function Ratios* is only computed to seven decimal places, we cannot trust the last column to more than seven decimals. It is thus not worth while including a_8. At a_6 we are correct to seven decimals. Thus when q is large compared to p, six terms are adequate to obtain seven-figure accuracy. When four-figure accuracy is adequate, a_4 will suffice. To obtain adequate values for the I_s's we used δ^4 interpolation for a_2 and a_3, but found that with a_3 the inclusion of δ^4, although modifying I_3 in the seventh decimal place, gave no change in the value of $a_3 I_3$ up to the ninth decimal place. It is accordingly sufficient to interpolate the I_s values, using only δ^2, except in the case of the values $s=2$, or perhaps $s=3$, as p increases relative to q.

It would thus seem that the method is satisfactory if p be as low as 4. Further, it can be a good deal shortened, if we do not use formulae (viii), but proceed in the following manner. We have seen by (v) that

$$c_s = \frac{s-1}{s}\, c_{s-1} - \frac{q}{s}\, c_{s-2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(v)},$$

and further, by (iv), that

$$a_s = c_s \frac{(p+s-1)(p+s-2)\ldots(p+1)\,p}{(q-1)^s} \frac{\Gamma(p+q)}{\Gamma(q)(q-1)^p}$$

$$= C_s \times f_s = C c_s \frac{(p+s-1)(p+s-2)\ldots(p+1)\,p}{(q-1)^s} \quad \ldots\ldots\ldots\ldots\text{(ix)},$$

where $C = \dfrac{\Gamma(p+q)}{\Gamma(q)(q-1)^p}$, and $f_s = c_s \dfrac{(p+s-1)(p+s-2)\ldots(p+1)\,p}{(q-1)^s}$.

Hence in (v), replacing c_s by a_s, we find

$$a_s = \frac{p+s-1}{(q-1)^s}\{(s-1)\,a_{s-1} - (p+s-2)\,a_{s-2}\} \quad \ldots\ldots\ldots\ldots\text{(x)},$$

or, again,

$$f_s = \frac{p+s-1}{(q-1)^s}\{(s-1)\,f_{s-1} - (p+s-2)\,f_{s-2}\} \quad \ldots\ldots\ldots\ldots\text{(x bis)}.$$

The latter formula has some advantages if we wish to deal with more moderate numbers, as C may be large. We then multiply by C at the end of our desired series of f_s's.

Now the series of a_s's or f_s's can be most rapidly ascertained in succession by the machine from (x) or (x *bis*). For example, $a_0 = C$, $a_1 = 0$, or $f_0 = 1$, $f_1 = 0$, and for $p = 10$, $q = 101$, say, we have the series

$$a_2 = \tfrac{11}{200}(a_1 - 10a_0), \quad a_3 = \tfrac{12}{300}(2a_2 - 11a_1), \quad a_4 = \tfrac{13}{400}(3a_3 - 12a_2),$$
$$a_5 = \tfrac{14}{500}(4a_4 - 13a_3), \quad a_6 = \tfrac{15}{600}(5a_5 - 14a_4), \quad \text{etc.}$$

The rapidity of this method is great, but it has the disadvantage that, if a slip be made in one a_s, it is carried on to later a's. Hence it is advisable to check the last a by means of the corresponding value of a in (viii). Thus there is no difficulty in finding the a's to the s needed to give the number of places necessary in the required incomplete B-function. In the example worked out above, we obtain seven decimal-place accuracy for $s = 6$.

Actually at the very best we cannot hope for greater accuracy, for the *Tables of the Incomplete* Γ-*Function* only reach to seven decimals, and accordingly, as C grows greater than unity, we shall not be correct in the sixth or seventh figure when we multiply by C. We need a Γ-function table to some 10 to 12 figures, if C lies between 300 and 400. We will illustrate these points on a couple of examples.

Illustration (ii). Let us take $p = 10$, $q = 101$ and $x_0 = \cdot 075$. Here

$$C = 1{\cdot}7018{,}2143{,}78,$$

and the general form for

$$I_s = I\left(\frac{x_0(q-1)}{\sqrt{p+s}}, p+s-1\right) = I\left(\frac{7{\cdot}5}{\sqrt{p+s}}, q+s\right),$$

and further

$$f_s = \frac{q+s}{160s}\{(s-1)f_{s-1} - (8+s)f_{s-2}\}.$$

Working the f's out in succession from this, we obtained the values given in the second column of the table below.

The true value of the function found by "brute force" integration and multiplying out was ·3109,71212, and checked by quadrature ·3109,7120. The value ·3109,712 will be correct to seven places and serve for comparison.

Table for $p = 10$, $q = 101$ *and* $x_0 = \cdot 075$.

	f_s	$a_s = C \times f_s$	I_s From Γ-Function Tables	$a_s I_s$	$S(a_s I_s)$ up to a_s
f_0	1·0000,0000	1·7018,2143,78	$I_0(2{\cdot}3717,0824{,}5,\ 9) = \cdot 2235{,}9236$	+·3805,1427	+·3805,1427
f_2	−·55	−·9360,0179,08	$I_2(2{\cdot}1650,6350{,}9,\ 11) = \cdot 0792{,}4131$	−·0741,7001	+·3063,4426
f_3	−·044	−·0748,8014,33	$I_3(2{\cdot}0801,2573{,}6,\ 12) = \cdot 0426{,}6583$	−·0031,9482	+·3031,4944
f_4	+·21021	+·3577,3988,44	$I_4(2{\cdot}0044,5931{,}4,\ 13) = \cdot 0215{,}6463$	+·0077,1453	+·3108,6397
f_5	+·0395,5952	+·0673,2323,92	$I_5(1{\cdot}9364,9167{,}3,\ 14) = \cdot 0102{,}6042$	+·0006,9076	+·3115,5473
f_6	−·0686,2856,00	−·1167,9355,47	$I_6(1{\cdot}875,\ 15) = \cdot 0046{,}0831$	−·0005,3822	+·3110,1651
f_7	−·0229,7518,08	−·0390,9965,52	$I_7(1{\cdot}8190,1718{,}8,\ 16) = \cdot 0019{,}5890$	−·0000,7659	+·3109,3992
f_8	+·0199,1615,23	+·0338,9373,49	$I_8(1{\cdot}7677,6695{,}3,\ 17) = \cdot 0007{,}9001$	+·0000,2678	+·3109,6670
f_9	+·0109,9814,58	+·0187,1688,03	$I_9(1{\cdot}7206,1800{,}4,\ 18) = \cdot 0003{,}0301$	+·0000,0567	+·3109,7237
f_{10}	−·0049,3064,11	−·0083,9107,07	$I_{10}(1{\cdot}6770,5098{,}3,\ 19) = \cdot 0001{,}1074$	−·0000,0093	+·3109,7144
f_{11}	−·0046,9583,97	−·0079,9148,07	$I_{11}(1{\cdot}6366,3417{,}7,\ 20) = \cdot 0000{,}3864$	−·0000,0031	+·3109,7113
f_{12}	+·0008,2177,53	+·0013,9851,48	$I_{12}(1{\cdot}5990,0537{,}3,\ 21) = \cdot 0000{,}1288$	+·0000,0002	+·3109,7115

The a's are correct to their ten decimals and there is no difficulty in tabling more, in fact a_{13}, a_{14}, a_{15} and a_{16} were found. But to make use of these a_s's it would be needful to have the I_s's to more than seven figures. The I_s's found by interpolation have been used to *eight* figures, but the last figure is, of course, unreliable. The value of C being >1, the eighth and ninth figures of I for the lower values of s would be needful to obtain the value of the incomplete B-function exact to the seventh decimal place. Thus the value obtained from the Γ-Function Tables is ·3109,7115 or six units in error in the eighth figure. This is what we might have anticipated. On the basis of this example we can assert that for $p=10$ the true value can be found to some six units in the eighth decimal.

Application of Dr Müller's Method.

In order to test the labour required in working various processes for finding the incomplete B-function, we worked our $I_{\cdot 075}$ (10, 101) by Müller's Continued Fraction Method*. The results are given in the table below. It may be noted that when two successive convergents agree to the number of decimal places we require, we may stop our computing, and take the required value as that common to the two convergents.

We shall use practically Dr Müller's notation*. The true value of

$$I_{\cdot 075}(10, 101) = \cdot 3109{,}7121\dagger,$$

$$k = p + q - 1 = 110, \quad y_0 = 1 - x_0 = \cdot 925, \quad C = \frac{x_0^{\,p}\, y_0^{\,q-1}\, \Gamma(k+1)}{\Gamma(p+1)\,\Gamma(q)} = \cdot 1086{,}2676{,}9752,$$

$$u_r = \frac{101 - r}{10 + r}, \quad t = \frac{x_0}{y_0} = \frac{3}{37}, \quad p_r = p_{r-1} + b_r\, p_{r-2}, \quad q_r = q_{r-1} + b_r\, q_{r-2}.$$

r	b_r	p_r	q_r	Cp_r/q_r = convergent to $I_{\cdot 075}(10, 101)$
1	1·0000,0000,00	1·0000,0000,00	1·0000,0000,00	—
2	− ·7371,0073,71	1·0000,0000,00	·2628,9926,29	—
3	+ ·0681,8181,82	1·0681,8181,82	·3310,8108,11	—
4	− ·5660,0831,60	·5021,7350,22	·1822,7791,20	—
5	+ ·0997,9209,98	·6087,6960,88	·2153,1718,83	—
6	− ·4540,5405,41	·3807,5569,43	·1325,5316,34	—
7	+ ·1145,2702,70	·4504,7626,77	·1572,1280,08	·3112,58254
8	− ·3758,9427,66	·3073,5238,14	·1073,8682,53	·3109,01233
9	+ ·1208,2670,91	·3617,8194,64	·1263,8233,06	·3109,54894
10	− ·3186,3442,39	·2638,4889,74	·0921,6519,14	·3109,74816
11	+ ·1226,8847,80	·3082,3537,38	·1076,7084,72	·3109,71947
12	− ·2750,9652,51	·2356,5145,90	·0823,1652,33	·3109,71063
13	+ ·1221,4812,21	·2733,0183,11	·0954,6831,51	·3109,71185
14	− ·2409,9989,32	·2165,0985,46	·0756,3004,18	·3109,71217

It will be seen that convergents 7 and 8 give three-figure accuracy, 9 and 10 four-figure accuracy, 10 and 11 five-figure accuracy, 12 and 13 six-figure accuracy,

* *Biometrika*, Vol. XXII. pp. 291 *et seq.* See also our first footnote, p. 423 below.

† Value obtained by "brute force," by quadrature and by the Γ-Function Table.

and 13 and 14 the required seven-figure accuracy. Müller's method is certainly shorter than the "brute force" or quadrature methods, and about on a par with the Γ-Function Table method at $p = 10$.

Illustration (iii). A still more extreme case was now taken, namely,

$$p = 36, \quad q = 101 \quad \text{and} \quad x_0 = \cdot 20.$$

Here $C = 392\cdot0702,0930$, and it is clear that with a seven-figure table only of the incomplete Γ-function we cannot hope after multiplying by such a C to reach more than about four-figure accuracy for an incomplete B-function.

We found the following values:

Table for $p = 36$, $q = 101$ *and* $x_0 = \cdot 20$.

	f_s	$a_s = Cf_s$	I_s Values from Γ-Tables	$a_s I_s$	$S(a_s I_s)$ up to $a_s I_s$
f_0	1·0000,0000	392·0702,0930	$I_0(3\cdot3333,3333,3,\ 35) = \cdot0008,0363$	+·3150,7938	+·3150,7938
f_1	− 6·66	− 2611·1875,9394	$I_1(3\cdot2444,2842,1,\ 37) = \cdot0002,1708$	−·5668,3660	−·2517,5722
f_2	− 1·6872	− 661·5008,5713	$I_2(3\cdot2025,6307,7,\ 38) = \cdot0001,0868$	−·0718,9191	−·3236,4913
f_3	+ 24·181,794	+ 9480·9610,3483	$I_3(3\cdot1622,7766,0,\ 39) = \cdot0000,5323$	+·5046,7156	+·1810,2243
f_4	+ 13·0022,3808	+ 5097·7902,0539	$I_4(3\cdot1234,7523,9,\ 40) = \cdot0000,2547$	+·1298,4072	+·3108,6315
f_5	− 61·6544,7226	− 24172·8818,4326	$I_5(3\cdot0860,6699,9,\ 41) = \cdot0000,1192$	−·2881,4075	+·0227,2240
f_6	− 54·1811,1569	− 21242·8013,6869	$I_6(3\cdot0499,7140,9,\ 42) = \cdot0000,0541$	−·1149,2356	−·0922,0116
f_7	+118·7993,2634	+ 46577·6767,4282	$I_7(3\cdot0151,1344,5,\ 43) = \cdot0000,0239$	+·1113,2065	+·0191,1949
f_8	+160·3644,8195	+ 62874·1360,0242	$I_8(2\cdot9814,2396,9,\ 44) = \cdot0000,0102$	+·0641,3162	+·0832,5111
f_9	−170·2750,5096	− 66759·7748,6846	$I_9(2\cdot9485,3912,2,\ 45) = \cdot0000,0045$	−·0300,4190	+·0532,0921
f_{10}	−372·9827,2825	−146235·4163,3026	$I_{10}(2\cdot9172,9982,6,\ 46) = \cdot0000,0022$	−·0321,7179	+·0210,3742
f_{11}	+146·0854,9140	+ 57275·7691,8889	$I_{11}(2\cdot8867,5134,6,\ 47) = \cdot0000,00105$	+·0060,1396	+·0270,5138
f_{12}	+711·9955,9844	+279152·2633,0105	$I_{12}(2\cdot8571,4285,7,\ 48) = \cdot0000,00050$	+·0139,5761	+·0410,0899
f_{13}					

The true value is ·0409,4983, or the excess at this point is roughly six units in the fifth figure. But even this agreement is illusory. It depends on taking the value of I_s to eight or nine figures in the last two cases, where our tables only give seven figures! Had we taken $I_{12} = \cdot0000,00104$ instead of ·0000,00105 we should have been only two units in error in the sixth figure instead of six in the fifth figure. But the reader will ask how we found I_{12} and I_{13}. If he will turn up $p = 47$ and the argument lying between 2·8 and 2·9 in the *Incomplete Γ-Function Tables*, he will find that he has to interpolate between

$$u_0 = \cdot0000,000, \quad \delta^2 u_0 = \cdot0000,000, \quad \delta^4 u_0 = \cdot0000,000, \text{ and}$$
$$u_1 = \cdot0000,001, \quad \delta^2 u_1 = \cdot0000,002, \quad \delta^4 u_1 = \cdot0000,001,$$

and matters are still worse for I_{13}. These units in the last figure may be anything from just over 05 to just under 15, and when we have to multiply by over 392, what they are is essential. Actually interpolating from the *Table*, the value found for I_{12} is ·0000,00080, which can only be taken as ·0000,001, but in multiplying by over 392, it is essential to know whether 80 or 100 is the better value to adopt. Actually, when the a_s's are as large as they are in this example, a table of the incomplete Γ-function ratio would have to exist to at least *ten* decimal places.

It is not the large number of a_s's required which really distresses us—they are not hard to run off on the machine. The difficulty is that the I_s's cannot be obtained to an adequate number of places. How then did we select I_{12} to be ·0000,00105 and I_{13} to be ·0000,00050? Simply by looking at the I_s column and noting that I_{s+1} is always somewhat less than half I_s, the preceding value. A round number *under* half 22 is 105, and a round number under half 105 is 50! It is clear that such a process is by no means to be commended, and taken together with the erratic values of the a_s's, so obvious in the products, $a_s I_s$, we must conclude that the reduction of an incomplete B-function to the *Tables of the Incomplete* Γ-*Function*, while satisfactory for $p = 10$ and $q = 101$, and probably some way beyond this value of p, fails to be a practically useful method when p gets to 36, or some third or more of q. We need, therefore, a more satisfactory method when p and q approach more nearly to equality, and one or both lie outside the B-Function Table.

Various methods have been applied, such as expanding the incomplete B-function in terms of incomplete B-functions lying inside the published *Incomplete* B-*Function Table*, but none of these seemed for p and q of the order of 36 and 101 as short as Dr Müller's continued fraction method. That method has peculiar advantages, when we only need the incomplete B-function ratio to four or five decimal places; it converges steadily from the early $C \times p_s/q_s$, while the $S(a_s I_s)$ may be most erratic before it approaches the true value.

In all the methods there are many chances of even a good computer making a slip. But systematic arrangement of the material and experience in the use of the methods makes them far shorter than appears at first sight. The reader must also remember that it is solely the mathematician who, for special purposes, will aim at seven-figure accuracy. For the statistician, four- or even three-figure accuracy is often adequate.

We give in the table below the data requisite to find $I_{\cdot 2}(36, 101)$ to seven-figure accuracy by continued fractions.

$$I_{\cdot 2}(36, 101) = \cdot 0409{,}4983 \text{ (by quadrature).}$$

$$p = 36, \quad q = 101, \quad k = p + q - 1 = 136, \quad x_0 = \cdot 2, \quad y_0 = 1 - x_0 = \cdot 8,$$

$$C = \frac{x_0^p y_0^{q-1} \Gamma(k+1)}{\Gamma(p+1)\,\Gamma(q)} = \cdot 0147{,}53914, \quad u_r = \frac{101 - r}{36 + r}, \quad t = \frac{x_0}{y_0} = \frac{1}{4},$$

$$p_r = p_{r-1} + b_r p_{r-2}, \quad q_r = q_{r-1} + b_r q_{r-2}{}^*.$$

Thus the eighth and ninth convergents give the value correct to four figures, the eleventh and twelfth correct to five figures, the thirteenth and fourteenth to seven figures as required.

* The general expressions for the b's are

$$b_{2r} = -\frac{(p+r-1)(p+r)}{(p+2r-2)(p+2r-1)} u_r t, \quad b_{2r+1} = \frac{r(k+r)}{(p+2r-1)(p+2r)} t,$$

while $$p_1 = q_1 = 1, \text{ and } p_2 = 1.$$

r	b_r	p_r	q_r	Continued fraction p_r/q_r	$C \times p_r/q_r$, converging to $I_{\cdot 2}(36, 101)$
1	+1·0000,0000,00	1·0000,0000,00	1·0000,0000,00	—	—
2	− ·6756,7567,57	1·0000,0000,00	·3243,2432,43	—	—
3	+ ·0243,5988,62	1·0243,5988,62	·3486,8421,05	—	—
4	− ·6179,1497,98	·4064,4490,64	·1482,7935,22	—	—
5	+ ·0442,3076,92	·4517,5313,21	·1637,0192,30	—	—
6	− ·5676,8292,68	·2210,2129,81	·0795,2626,64	2·7792,2386,45	·0410,0443
7	+ ·0605,4006,97	·2483,7046,42	·0894,3679,22	2·7770,5022,83	·0409,7236
8	− ·5236,7109,63	·1326,2799,87	·0477,9118,51	2·7751,5609,67	·0409,4441
9	+ ·0739,9577,17	·1510,0636,29	·0544,0912,96	2·7753,8648,40	·0409,4789
10	− ·4848,4848,48	·0867,0187,87	·0312,3764,59	2·7755,5738,28	·0409,5033
11	+ ·0851,4492,75	·0995,5930,45	·0358,7030,73	2·7755,3531,02	·0409,5001
12	− ·4503,9315,45	·0605,0937,19	·0218,0108,54	2·7755,2807,11	·0409,5127
13	+ ·0944,1489,36	·0699,0925,30	·0251,8777,66	2·7755,2298,92	·0409,4983
14	− ·4196,4285,71	·0455,1692,73	·0160,3910,68	2·7755,2408,97	·0409,4984

Conclusions. Methods have been indicated of throwing back values of the incomplete Γ-function ratio and the incomplete B-function ratio lying outside the ranges of the published tables on the *Tables of the Incomplete* Γ-*Function.* These methods work adequately for the Γ-function and also adequately for the B-function, if p be small as compared with q. But in the latter case, if p be comparable with q, the constant $C = \dfrac{\Gamma(p+q)}{\Gamma(q)\,(q-1)^p}$ becomes very large, and the existing tables of the Γ-function ratio only giving seven decimal places are inadequate for the method, which would require at least ten decimal places. After considering various methods for determining an incomplete B-function ratio with comparable p and q, we believe the most efficient to be Müller's method of the continued fraction, illustrations of which are given in the paper.

The authors of this paper have not overlooked the value of the late Mr Soper's tract, *Numerical Evaluations of the Incomplete* B-*Function**, but have endeavoured to make a somewhat more direct answer to the question often put: What is to be done when we need the numerical value of incomplete Eulerian integrals lying outside the existing tables?

In conclusion we desire cordially to thank Miss F. N. David for kindly working out certain values for us, as, being away from home, we had no machine, and no table of more than seven-figure logarithms.

* *Tracts for Computers*, No. VII. Cambridge University Press.

FURTHER INVESTIGATION OF THE MORPHOMETRIC CHARACTERS OF THE INDIVIDUAL BONES OF THE HUMAN SKULL.

By KARL PEARSON and T. L. WOO, Ph.D.

(1) *Introduction.*

In 1930 the then Director of the Biometric Laboratory determined that the time was ripe to undertake a wide system of measurements on the individual bones of the human skull. A series of 63 measurements was selected, which could be taken on the non-disjointed crania, and the term morphometric was introduced for such measurements to distinguish them from the existing ethnometric measurements, which may cover several bones at once. It was noted, however, that morphometric measurements may be found to have first-class ethnographical value when, in the future, they shall have been taken on a number of racial groups. It was proposed to start the morphometric measurements on an adequately long series of crania, and discuss in the first place the results which flowed from the measurement of homologous bones. The well-known E series of Egyptian crania of the 26th to the 30th dynasties, numbering about 800 male crania, were chosen, and an indefatigable worker, both from the measuring and computing sides, Dr T. L. Woo, was selected to undertake the work in 1930. After a year's labour in measuring and computing, Dr Woo published, in 1931*, the first report on these morphometric measurements, with some assistance from the Director of the Laboratory in the preparation of his text. This dealt with the 50 characters out of the 63, which belonged to homologous bones, i.e. 25 pairs of corresponding measurements on the right and left sides of the skull. The whole report dealt with absolute sizes, and not with the shapes of individual bones. The discussion was chiefly concerned with asymmetry in size. The following conclusions were reached:

(i) The mean or typ skull of a race is definitely and markedly asymmetrical; asymmetry is not only a peculiarity of the individual skull; it is also a racial character.

(ii) The frontal and parietal bones are with marked significance greater on the right side, so also are the approximately antero-posterior lengths on the temporal. Both malar measurements are in excess on the left, but these measurements have little to do with the brain. On the other hand while the arcs from asterion to sagittal plane†, and from asterion to basion were significantly greater on the right, that from the lambda to the asterion along the lambdoid suture was *markedly* dominant on the left.

* *Biometrika*, Vol. xxii. pp. 324—352.

† For definition of this (and other) measurements see *ibid.* pp. 325—329.

AST=asterion
AUR=auriculare
POR=Martin's Porion
β =bregma
K =krotaphion
λ =lambda
O =opisthyon
PM =post-molary point
SN =superior nasal point
N =nasion
SP =sphenion
ST =stephanion
ZM=zygomaxillare
arcs
chords

Diagram 1

(iii) The variability, either absolute or relative, showed no marked lateral trend, and the greater lateral variability was not significantly associated with the greater lateral characters.

The report showed the large part played by homology in the correlation of morphometric characters. It also indicated that facial and frontal homologous characters were more highly correlated (·86—·98) while those of the temporal, parietal and occipital regions were less so (·54—·85). The face, possibly owing to sexual selection, is the most symmetrical region of the skull. This report, however, left much for future consideration and the present paper will deal with many problems left unanswered in the first report.

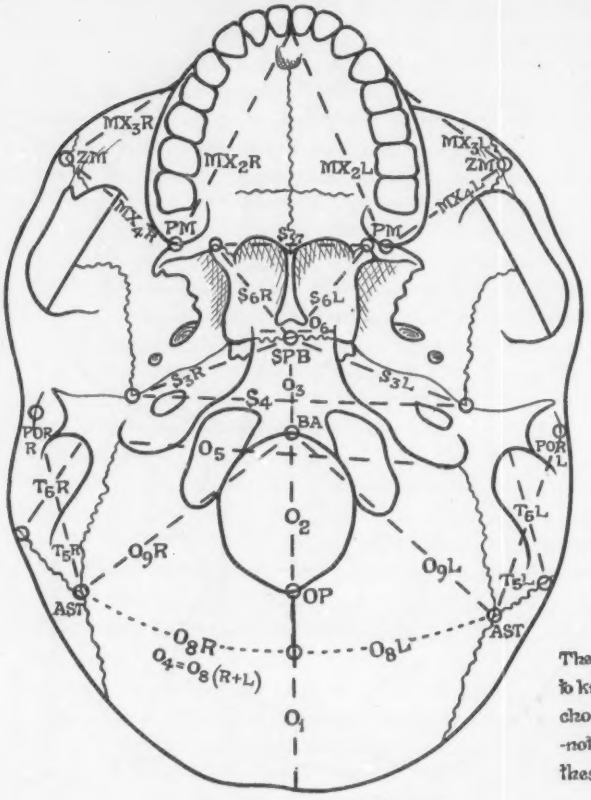

Diagram 2.

The chord from krotaphion to krotaphion S_1, and the chords $S_5(R)$ and $S_5(L)$ cannot be properly shown on these diagrams.

Diagrams 1 and 2, giving respectively the norma lateralis and norma basalis of a skull, indicate approximately the measurements taken during the whole of this investigation.

(2) *Constants of the Thirteen Additional Characters.*

We place first on record the remaining thirteen characters dealt with in Dr Woo's measurements; these were principally but not entirely taken along or perpendicular to the median sagittal plane. We shall not repeat here the definitions of the 50 bilateral characters discussed in the first report, but we reproduce here the diagrams indicating those characters with addition of the thirteen further characters.

List of the Thirteen Additional Characters.

Frontal
- F_3, Minimum arc between the terminals of the minimum frontal diameter. $\{B'\}$.
- F_4, Minimum arc from nasion to bregma. $\{S_1\}$.
- F_5, Minimum arc from right stephanion to left stephanion.

Parietal
- P_1, Minimum arc from bregma to lambda. $\{S_2\}$.

Occipital
- O_1, Minimum arc from lambda to opisthion. $\{S_3\}$.
- O_2, Chord from opisthion to basion. $\{fml\}$.
- O_3, Chord from basion to spheno-basion.
- O_4, Minimum arc from right asterion to left asterion.
- O_5, Chord between the points, right and left, where the occipito-mastoid suture meets the posterior border of the jugular foramen.
- O_6, Breadth of the pars basilaris, taken between the legs of small calipers as far forward as possible.

Sphenoid
- S_1, Chord between right and left krotaphions.
- S_4, Chord between the extreme posterior points on the right and left of the sphenoid exposed on the base of the skull.
- S_7, Chord between the lowest points, right and left on the suture between the medial pterygoid plate and the palate bone.

Four of these are familiar ethnometric characters and we have placed against them in curled brackets the symbols used to denote them in the Biometric Laboratory memoirs. They are the sole cranial characters common to the present morphometric and the earlier ethnometric researches of that Laboratory.

We have tabled the constants of these thirteen additional characters in Table I, p. 428.

(3) *Variation of the Sixty-three Characters.*

We are now able to consider the variation of our 63 characters taken as a single group. In Table II (p. 429), we have collected the variation constants: in column (*a*) the bone; in column (*b*) its character; in column (*c*) the absolute variation of the character; in column (*d*) the relative variation, or percentage variation on size of character; in column (*e*) the characters of the bone in order of intensity of absolute variation; in column (*f*) the order of intensity of relative variation, are provided. In the last two columns the order of absolute sizes of the characters are placed for each bone character in square brackets. When we examine column (*e*) we see that

TABLE I. *Constants of the Non-Bilateral Characters used in this Paper.*

(This Table must be taken in conjunction with Table I of the former Memoir*.)

Bone	Character (cases)	Means	Standard Deviations	Coefficients of Variation
Frontal	F_3 (891)	104·5839 ± ·1447	6·4047 ± ·1023	6·1240 ± ·0982
	F_4 (887)	127·2100 ± ·1413	6·2372 ± ·0999	4·9031 ± ·0787
	F_5 (888)	139·5224 ± ·2181	9·6355 ± ·1542	6·9061 ± ·1110
Parietal	P_1 (887)	128·3726 ± ·1651	7·2475 ± ·1167	5·6456 ± ·0912
Occipital	O_1 (864)	115·0093 ± ·1574	6·8604 ± ·1113	5·9651 ± ·0972
	O_2 (882)	35·7330 ± ·0592	2·6047 ± ·0418	7·2894 ± ·1177
	O_3 (870)	26·3397 ± ·0534	2·3343 ± ·0377	8·8625 ± ·1444
	O_4 (875)	128·0791 ± ·1378	6·0422 ± ·0974	4·7175 ± ·0762
	O_5 (874)	73·2154 ± ·0818	3·5874 ± ·0579	4·8998 ± ·0792
	O_6 (864)	21·8594 ± ·0418	1·8199 ± ·0295	8·3256 ± ·1362
Sphenoid	S_1 (732)	114·5027 ± ·1332	5·3447 ± ·0942	4·6677 ± ·0825
	S_4 (881)	70·3871 ± ·0817	3·5943 ± ·0578	5·1065 ± ·0823
	S_7 (822)	35·4720 ± ·0687	2·9187 ± ·0486	8·2281 ± ·1378

the absolute variations in the characters of each individual bone, even allowing for random sampling, are not proportional to the absolute size of the organs measured. Thus it is demonstrated that the coefficients of variation will not be constant—any more than the standard deviations for a single cranial bone, much less for all the bones, i.e. the coefficients of variation must depend on the size of the character measured.

At the same time it will be clear that there is considerable positive correlation between the intensity of the absolute variation and the size of the varying character.

This is shown by our Table II where the intensities of the variation are arranged in decreasing order, and against them are put the sizes of the corresponding organs, [1] marking the largest measurement, [2] the second and so on for each bone. If we now average the bracket numbers for the upper and lower halves of the group of characters for each bone, we find that for the six bones—frontal, temporal, sphenoid, malar, maxillary and occipital—these averages are much lower for the greater intensities of variation, only in the case of the parietal are they equal. In other words, the absolute variabilities of the larger-sized characters are the larger.

The last column (f) shows, however, that the coefficient of variation overcorrects this defect; for if we except the frontal where equality exists, and the short malar series of only four characters, where the two larger coefficients of variation belong to the two large-sized characters, then in the case of all the remaining five bones

* *Biometrika*, Vol. XXII. pp. 330—331.

TABLE II. *Variability, Absolute and Relative, of the 63 Morphometric Characters.*

(a) Bone	(b) Character	(c) Standard Deviation	(d) Coefficient of Variation	(e) Order of Absolute Variation	(f) Order of Relative Variation
Frontal	F_1 R	5·10	5·18	F_5 [1] Mean First	F_5 [1] Mean First
	F_1 L	5·06	5·16	F_3 [3] Four *2·50*	F_3 [3] Four *4·25*
	F_2 R	4·78	5·43	F_4 [2]	$F_2(L)$ [7]
	F_2 L	4·98	5·69	$F_1(R)$ [4]	$F_2(R)$ [6]
	F_3	6·40	6·12	$F_1(L)$ [5] Mean Last	$F_1(R)$ [4] Mean Last
	F_4	6·24	4·90	$F_2(L)$ [7] Four *5·50*	$F_1(L)$ [5] Four *4·25*
	F_5	9·64	6·91	$F_2(R)$ [6]	F_4 [2]
Parietal	P_1	7·25	5·65	P_1 [5] Mean First	P_1 [5] Mean First
	P_2 R	5·93	5·27	$P_3(L)$ [4] Four *4·25*	$P_2(R)$ [6] Four *5·50*
	P_2 L	5·70	5·13	$P_4(L)$ [2]	$P_2(L)$ [7]
	P_3 R	5·92	3·58	$P_2(R)$ [6]	$P_3(L)$ [4]
	P_3 L	6·04	3·70	$P_3(R)$ [3] Mean Last	$P_3(R)$ [3] Mean Last
	P_4 R	5·84	3·30	$P_4(R)$ [1] Four *4·25*	$P_4(L)$ [2] Four *2·50*
	P_4 L	5·96	3·39	$P_2(L)$ [7]	$P_4(R)$ [1]
Temporal	T_1 R	3·89	4·48	$T_4(L)$ [2] Mean First	$T_6(R)$ [12] Mean First
	T_1 L	3·83	4·46	$T_4(R)$ [1] Seven *4·71*	$T_6(L)$ [11] Seven *10·14*
	T_2 R	3·78	8·11	$T_5(R)$ [5]	$T_7(R)$ [13]
	T_2 L	3·71	7·93	$T_3(L)$ [6]	$T_3(R)$ [8]
	T_3 R	4·27	6·45	$T_1(R)$ [3]	$T_7(L)$ [14]
	T_3 L	4·25	6·47	$T_1(L)$ [4]	$T_2(L)$ [7]
	T_4 R	4·30	4·31	$T_6(R)$ [12]	$T_3(L)$ [6]
	T_4 L	4·48	4·50	$T_6(L)$ [11] Mean Last	$T_3(R)$ [5] Mean Last
	T_5 R	2·75	5·98	$T_2(R)$ [8] Seven *10·29*	$T_5(L)$ [9] Seven *4·71*
	T_5 L	2·77	6·01	$T_2(L)$ [7]	$T_5(R)$ [10]
	T_6 R	3·79	8·33	$T_7(R)$ [13]	$T_4(L)$ [2]
	T_6 L	3·78	8·24	$T_7(L)$ [14]	$T_1(R)$ [3]
	T_7 R	2·94	8·14	$T_5(L)$ [9]	$T_1(L)$ [4]
	T_7 L	2·86	7·94	$T_5(R)$ [10]	$T_4(R)$ [1]
Sphenoid	S_1	5·34	4·67	S_1 [1] Mean First	S_7 [11] Mean First
	S_2 R	3·43	4·57	S_4 [4] Six *3·50*	$S_6(L)$ [10] Six *8·00*
	S_2 L	3·50	4·68	$S_2(L)$ [3]	$S_6(R)$ [9]
	S_3 R	1·93	5·33	$S_5(L)$ [6]	$S_5(L)$ [6]
	S_3 L	2·01	5·51	$S_5(R)$ [5]	$S_5(R)$ [5]
	S_4	3·59	5·11	$S_2(R)$ [2]	$S_3(L)$ [7]
	S_5 R	3·44	6·07	S_7 [11] Mean Last	$S_3(R)$ [8] Mean Last
	S_5 L	3·45	6·12	$S_6(L)$ [10] Six *7·83*	S_1 [4] Six *4·86*
	S_6 R	2·40	6·74	$S_6(R)$ [9]	$S_2(L)$ [3]
	S_6 L	2·44	6·87	$S_3(L)$ [7]	S_1 [1]
	S_7	2·92	8·23	$S_3(R)$ [8]	$S_2(R)$ [2]
Malar	Ml_1* R	4·38	7·37	$Ml_1(L)$ [1] Mean First	$Ml_1(L)$ [1] Mean First
	Ml_1* L	4·55	7·64	$Ml_1(R)$ [2] Two *1·50*	$Ml_1(R)$ [2] Two *1·50*
	Ml_2* R	3·11	6·30	$Ml_2(L)$ [3] Mean Last	$Ml_2(L)$ [3] Mean Last
	Ml_2* L	3·21	6·42	$Ml_2(R)$ [4] Two *3·50*	$Ml_2(R)$ [4] Two *3·50*
Maxillary	Mx_1 R	3·93	5·90	$Mx_1(R)$ [1] Mean First	$Mx_3(L)$ [5] Mean First
	Mx_1 L	3·90	5·88	$Mx_1(L)$ [2] Four *3·50*	$Mx_3(R)$ [6] Four *6·50*
	Mx_2 R	2·87	5·15	$Mx_3(L)$ [5]	$Mx_4(L)$ [8]
	Mx_2 L	2·89	5·17	$Mx_3(R)$ [6]	$Mx_4(R)$ [7]
	Mx_3 R	3·14	7·69	$Mx_2(L)$ [3] Mean Last	$Mx_1(R)$ [1] Mean Last
	Mx_3 L	3·25	7·94	$Mx_2(R)$ [4] Four *5·50*	$Mx_1(L)$ [2] Four *2·50*
	Mx_4 R	2·59	6·75	$Mx_4(L)$ [8]	$Mx_2(L)$ [3]
	Mx_4 L	2·68	6·99	$Mx_4(R)$ [7]	$Mx_2(R)$ [4]
Occipital	O_1	6·86	5·97	O_1 [2] Mean First	O_3 [11] Mean First
	O_2	2·60	7·29	O_4 [1] Six *4·17*	O_6 [12] Six *7·67*
	O_3	2·33	8·86	$O_7(L)$ [3]	O_2 [10]
	O_4	6·04	4·72	$O_7(R)$ [4]	O_1 [2]
	O_5	3·59	4·90	$O_8(R)$ [8]	$O_8(R)$ [8]
	O_6	1·82	8·33	O_5 [7]	$O_7(L)$ [3]
	O_7 R	5·34	5·49	$O_9(R)$ [5] Mean Last	$O_7(R)$ [4] Mean Last
	O_7 L	5·54	5·61	$O_9(L)$ [6] Six *8·83*	$O_8(L)$ [9] Six *5·33*
	O_8 R	3·69	5·78	$O_8(L)$ [9]	O_5 [7]
	O_8 L	3·29	5·20	O_2 [10]	O_4 [1]
	O_9 R	3·43	4·60	O_3 [11]	$O_9(R)$ [5]
	O_9 L	3·30	4·43	O_6 [12]	$O_9(L)$ [6]

* Unfortunately in Dr Woo's paper (*Biometrika*, Vol. XXII. p. 330, Table I, Ml_1 and Ml_2 should be interchanged; all the constants of the former are larger than those of the latter.

the moiety of the coefficients which are larger give a larger average of the square bracket numbers or correspond to characters of lesser size. In other words, the coefficient of variation of a character is not the same for all characters but roughly varies inversely as its size.

These results appear to indicate that if variation in the rates of bone growth be different from individual to individual, these rates cannot be the same for all directions of growth in the same bone. In other words, a bone in its growth must change not only its size but its shape. If for an individual, the individual bone changed its size only, not its shape, then it seems reasonable to suppose that the coefficients of variation would be the same for all measurements on the same bone. It seems then highly improbable that bones grow isotropically, or if growth from a centre of ossification be isotropic, the presence of more than one centre causes the growth of the bone as a whole to be heterotropic.

We can obtain some rough appreciation of the influence of absolute size on absolute variation by calculating the correlation coefficient ρ of order of size and order of variation for each bone and pooling these orders. We find that $\rho = \cdot 2687$ and, by the formula $r = 2 \sin (\frac{1}{6}\pi\rho)$, deduce

$$r = \cdot 2804.$$

This average correlation for the seven bones is a rough measure of the influence of the size of a character on its absolute variability; it is not so large as we might have anticipated. It would have been greater had we pooled all the place measurements together, but it seemed desirable to free ourselves of such great differences of size as occur, for example, between our sphenoidal and parietal measurements.

If the same method of correlation be applied to the absolute sizes and the coefficients of variation we find $\rho = -\cdot 1138$ and $r = -\cdot 1191$. Thus the relative variation exhibits over 50 % less correlation than the absolute variation does with absolute size, but it has overcorrected.

To sum up: The correlations of absolute size with both absolute and relative variability as represented by the standard deviation and the coefficient of variation respectively are small but significant in the case of measurements taken on the individual bones of the human skull. The correlation of absolute size with the standard deviation is positive, and with the coefficient of variation is negative; the latter is considerably smaller than the former, but the distribution of the total variation over the mean length overcorrects. So far no one has suggested a constant which will give a better measure of relative variation than the coefficient of variation*. It may be that a better exists, but it is more probable that variation is so peculiar to each character that it is not possible by the examination of a single

* It might be thought that a better coefficient could be found by taking the ratio of each deviation to its own length. Let c be the character, $\bar{c}$ its mean, σ_c its standard deviation, $v = \sigma_c/\bar{c}$ and $V = 100v$ its coefficient of variation. Then instead of $V = 100\sigma_c/\bar{c}$ let us take

$$F = 100 \sqrt{S \frac{(c-\bar{c})^2}{Nc^2}},$$

constant to assert whether A is more variable than B. Only when we compare the coefficients of variation of the same character in two sexes or two races can we get a partial answer to the question.

It remains to be considered whether it is possible to discover whether there are any special directions of measurement on the bones of the skull, transverse, sagittal or vertical, which are associated with larger relative variabilities. The difficulty here is that some of the measurements, especially some of the arcs, have very mixed directions.

The following classes were made:

Sagittal, *s*, i.e. practically parallel to the Frankfurt horizontal and the median sagittal plane. *st*, sagittal transverse, in which were included *t*(*s*), transverse, with a smaller amount of sagittal in it, and *s*(*t*), sagittal, with a smaller amount of transverse in it; *sv*, sagittal vertical, with which were included *s*(*v*) and *v*(*s*), sagittal with a less amount of vertical and vertical with some small amount of sagittal; and finally *svt*, a measurement combining all three directions. Thus for sagittal measurements we have *s*, *st*, *sv*, *stv*. In a similar manner we have for transverse measurements *t*, *ts*, *tv* and *tvs*, and for vertical measurements *v*, *vs*, *vt* and *vst*. The order of the letters in these classes is naturally indifferent. One or other of these classes was to the best of the writer's judgment affixed to every one of the 63 characters; and then the position of the coefficient of variation in each bone class taken and placed under the direction class of its measurement. These

where N is the total number of observations. Then

$$F^2=\frac{100^2}{N}\,S\,\frac{(\Delta c)^2}{(\bar{c}+\Delta c)^2},\text{ where }\Delta c=c-\bar{c}$$

$$=\frac{100^2}{N}\,S\left(\frac{\Delta c}{\bar{c}}\right)^2\left(1+\frac{\Delta c}{\bar{c}}\right)^{-2}$$

$$=\frac{100^2}{N}\,S\,\frac{(\Delta c)^2}{\bar{c}^2}\left(1-2\,\frac{\Delta c}{\bar{c}}+3\,\frac{(\Delta c)^2}{\bar{c}^2}-4\,\frac{(\Delta c)^3}{\bar{c}^3}+5\,\frac{(\Delta c)^4}{\bar{c}^4}-\ldots\right)$$

$$=100^2\left(\frac{\sigma_c^2}{\bar{c}^2}-2\sqrt{\beta_1}\,\frac{\sigma_c^3}{\bar{c}^3}+3\beta_2\,\frac{\sigma_c^4}{\bar{c}^4}-4\,\frac{\beta_5}{\sqrt{\beta_1}}\,\frac{\sigma_c^5}{\bar{c}^5}+5\beta_4\,\frac{\sigma_c^6}{\bar{c}^6}-\ldots\right),$$

where the β's have the customary statistical significance. For a symmetrical system $\sqrt{\beta_1}$ and $\beta_5/\sqrt{\beta_1}$ etc. $=0$, and for a normal distribution $\beta_2=3$, $\beta_4=15$, and then

$$F=100v\,(1+9v^2+75v^4)^{\frac{1}{2}},$$

if we neglect terms in v^6. But to the same degree of approximation

$$1+9v^2+75v^4=\left(1-\frac{23}{3}\,v^2\right)^{-\frac{27}{46}},$$

or, finally,

$$F=V\,\frac{1}{\left(1-\dfrac{23}{30{,}000}\,V^2\right)^{\frac{27}{46}}}.$$

Thus, though F may be theoretically more satisfactory than V, it is clear that F can only be constant as V becomes constant, and V is easier to compute than F. Indeed it is difficult to think of any other lengths than $\bar{c}$ and σ_c for a normal frequency which could give a ratio possibly independent of absolute size. It may be noted that when V is small, F will be almost identical with V in value, and that when V is even as high as six, F will only differ from V by about 1 to 2 per cent.

positions were then averaged for each class. As the positions denote descending order of intensity of relative variation, a low average position (a great number for the mean position) marks *less* variation. There resulted the following scheme:

(i) *Sagittal or in part sagittal measurements.*

Class ...	*s*	*st*	*sv*	*stv*	All
Mean Position	7·333	5·692	5·471	6·111	5·909

(ii) *Transverse or in part transverse measurements.*

Class ...	*t*	*ts*	*tv*	*tvs*	All
Mean Position	6·000	5·692	4·833	6·111	5·688

(iii) *Vertical or in part vertical measurements.*

Class ...	*v*	*vt*	*vs*	*vst*	All
Mean Position	3·875	4·833	5·471	6·111	5·179

This tabulation gives a uniformity of results far more decisive than anything which might have been anticipated. Whether we judge by the pure directions, *s*, *t*, *v*, or the total of measurements in which one of these directions appears, the order is the same; sagittal measurements are on the average least variable, vertical measurements most variable, and the transverse intermediate 7·333—6·000—3·875, or 5·909—5·688—5·179. And again when we have a bi-directional class *st* is less variable than *sv*, and *ts* less variable than *tv*. Again in group (iii) *v*, the most variable direction, is more variable when combined with *t* than when combined with *s*, because *t* is more variable than *s*.

These results seem undoubtedly suggestive; the individual bones of the skull tend to vary most in the vertical, less in the transverse and least in the sagittal direction.

If we venture to attribute individual deviations from the type to individual rates of growth, then the want of constancy in the coefficient of variation may be due to the fact that the growth of a cranial bone is not the same in the three fundamental directions, sagittal, transverse and vertical. Thus we may conclude that the coefficient of variation is likely to be most useful when we are comparing the *same* measurement in two races or in the two sexes.

If we attempt the same method on the absolute variations, even after "placing" by the separate bones and not by the skull as a whole, we do not get the same clear-cut result, the correlation between absolute size and absolute variation is too great. We find:

(i) *Transverse or in part transverse measurements.*

Class ...	*t*	*ts*	*tv*	*tvs*	All
Mean Position	5·600	6·000	4·667	3·833	5·233

(ii) *Sagittal or in part sagittal measurements.*

Class ...	*s*	*st*	*sv*	*stv*	All
Mean Position	5·167	6·000	6·684	3·833	5·886

(iii) *Vertical or in part vertical measurements.*

Class ...	*v*	*vt*	*vs*	*vst*	All
Mean Position	4·500	4·667	6·684	3·833	5·487

Thus, if judged by pure vertical lengths, the absolute variation is greater in the vertical than in the other two directions, but stands only second if we judge by all measurements which have a vertical element. Again the sagittal length if judged by the pure sagittal lengths is more variable than the transverse, but less variable if judged by all the measurements which have a sagittal element. In neither case do we get the same uniformity in the bi-directional groups as when we deal with the relative variabilities. Thus there appears to be a more systematic relationship among the relative than among the absolute variabilities. There is no full clue at present to the origin of the differences among the latter.

Thus judged by uni-directional averages the order of increasing absolute variability is transverse, sagittal and vertical, but judged by all variabilities in which these directions occur it is sagittal, vertical and transverse. Nor are the bi-directional place averages arranged in any satisfactory system. Accordingly, while the relative variabilities do fit a more or less consistent arrangement, the absolute variabilities beyond a moderate correlation with absolute size do not provide us with any clue at present to the source of the differences, and the apparent (if possibly not real) irregularities in their values.

(4) *On the Indices of Individual Cranial Bones.*

The large number of indices which could be formed, namely: 21 for the frontal, 21 for the parietal, 91 for the temporal, 55 for the sphenoid, 6 for the malar, 28 for the maxillary and 66 for the occipital bones, or a total of 288, rendered some selection needful. First, some 29 indices were chosen and the correlation tables formed. Among these were twelve pairs of indices for homologous characters and five for non-bilateral indices. These indices, their means and standard deviations, are given in Table III, p. 434, while in the last column of the table are placed the correlations of the corresponding indices of homologous bones.

We may first inquire whether the size of the index has any influence on its variation. In Table IV, p. 435, these indices are arranged in order of size, and we shall discuss their variabilities from this standpoint.

If we take the first 15, or greater indices, we find an average standard deviation of 5·96, and the last 15, or lesser indices, one of 5·58, on the basis of which we can make no definite statement. This view is confirmed if we remove the five non-bilateral indices which give a mean of 7·26, and average the first 12 homologous indices giving 5·68 and the last 12 giving 5·30. Again, if we compare the 12 pairs of homologous indices, the variations in the pairs are very close; but in six cases the larger and in six cases the less are associated with the greater variation. In view of these facts it is impossible to suppose that size of index is at all associated with intensity of variation of index. The coefficient of

TABLE III. *Constants of the Individual Bone Indices.*

Bone	Index and Number	Mean	Standard Deviation	Correlation of corresponding indices of homologous bones
Frontal	$100\,F_3/F_4$ (886)	82·4187 ± ·1216	5·3639 ± ·0859	—
	$100\,F_5/F_4$ (886)	109·7822 ± ·1591	7·0207 ± ·1125	—
	$100\,F_2/F_1$ (883) *R*	89·4779 ± ·0903	3·9797 ± ·0639	+ ·8171 ± ·0075
	$100\,F_2/F_1$ (883) *L*	89·3885 ± ·0943	4·1531 ± ·0667	
Parietal	$100\,P_2/P_1$ (743) *R*	88·1682 ± ·1472	5·9507 ± ·1041	+ ·7892 ± ·0093
	$100\,P_2/P_1$ (743) *L*	87·0283 ± ·1389	5·6125 ± ·0982	
	$100\,P_3/P_4$ (737) *R*	93·3989 ± ·0750	3·0181 ± ·0530	+ ·4057 ± ·0207
	$100\,P_3/P_4$ (737) *L*	92·9688 ± ·0760	3·0595 ± ·0538	
Temporal	$100\,T_2/T_3$ (843) *R*	70·5635 ± ·1446	6·2254 ± ·1023	+ ·7934 ± ·0086
	$100\,T_2/T_3$ (843) *L*	71·3737 ± ·1442	6·2058 ± ·1020	
	$100\,T_6/T_5$ (825) *R*	99·3649 ± ·1913	8·1484 ± ·1353	+ ·6185 ± ·0145
	$100\,T_6/T_5$ (825) *L*	100·1915 ± ·1909	8·1305 ± ·1350	
Sphenoid	$100\,S_5/S_4$ (723) *R*	80·7510 ± ·1603	6·3895 ± ·1134	+ ·8822 ± ·0056
	$100\,S_5/S_4$ (723) *L*	80·3195 ± ·1621	6·4642 ± ·1147	
	$100\,S_6/S_7$ (812) *R*	100·6478 ± ·1834	7·7478 ± ·1297	+ ·9048 ± ·0043
	$100\,S_6/S_7$ (812) *L*	100·5961 ± ·1858	7·8508 ± ·1314	
Malar	$100\,Ml_2/Ml_1$ (716) *R*	83·1522 ± ·1475	5·8490 ± ·1042	+ ·8806 ± ·0057
	$100\,Ml_2/Ml_1$ (716) *L*	83·9902 ± ·1518	6·0232 ± ·1073	
Maxillary	$100\,Mx_2/Mx_1$ (508) *R*	83·6063 ± ·1614	5·3937 ± ·1141	+ ·9344 ± ·0038
	$100\,Mx_2/Mx_1$ (508) *L*	84·2008 ± ·1622	5·4193 ± ·1147	
	$100\,Mx_3/Mx_1$ (414) *R*	61·1908 ± ·1101	3·3215 ± ·0779	+ ·7894 ± ·0125
	$100\,Mx_3/Mx_1$ (414) *L*	61·3768 ± ·1133	3·4180 ± ·0801	
Occipital	$100\,O_3/O_2$ (857)	83·4959 ± ·2209	9·5883 ± ·1562	—
	$100\,O_4/O_1$ (850)	110·8271 ± ·1623	7·0150 ± ·1148	—
	$100\,O_5/(O_2+O_3)$ (847)	117·8642 ± ·1674	7·2216 ± ·1184	—
	$100\,O_8/O_7$ (855) *R*	65·6947 ± ·1060	4·5929 ± ·0749	+ ·6044 ± ·0146
	$100\,O_8/O_7$ (855) *L*	64·4655 ± ·1000	4·3342 ± ·0707	
	$100\,O_8/O_9$ (849) *R*	85·6066 ± ·1242	5·3660 ± ·0878	+ ·4769 ± ·0179
	$100\,O_8/O_9$ (849) *L*	85·0942 ± ·1187	5·1271 ± ·0839	

variation, if indeed it had meaning in the case of indices, would thus not be steadier than the absolute variation as measured by the standard deviation. The cause or causes of such diversity of variation ranging from 3·0 to 9·6 units must be sought elsewhere than in size of index*.

* Dr G. M. Morant suggests that the order of laying down the bones and the period during which they grow—depending to some extent on their size—may provide a partial explanation of the diversity in variability and in the intensity of the effect of homology in the case of these indices. He points out that Mx_2, which is an indirect measure of the dental arcade, is probably determined at an early age,

Let us look into two further questions which can be answered from Table III.

The mean of the indices in Table III on the right side of the skull is 83·47, and that of the indices on the left side is 83·42, whence it is clear that there is no lateral dominance of the indices as a whole. Pursuing the same simple method, we find

Mean standard deviation on the right side = 5·499,

and Mean standard deviation on the left side = 5·483.

There is thus clearly no lateral dominance in variation as a whole. Thus by the use of indices we have largely freed ourselves from lateral dominance, either in size of index or magnitude of its variation.

TABLE IV. *Twenty-nine Indices of Cranial Bones in order of Size.*

Order	Index	Standard Deviation	Order	Index	Standard Deviation
1	$100\,O_5/(O_2+O_3)$	7·22	16	$100\,Mx_2/Mx_1\,(L)$	5·42
2	$100\,O_4/O_1$	7·02	17	$100\,Mx_2/Mx_1\,(R)$	5·39
3	$100\,F_5/F_4$	7·02	18	$100\,Ml_2/Ml_1\,(L)$	6·02
4	$100\,S_6/S_7\,(R)$	7·75	19	$100\,O_6/O_3$	9·59
5	$100\,S_6/S_7\,(L)$	7·85	20	$100\,Ml_2/Ml_1\,(R)$	5·85
6	$100\,T_6/T_5\,(L)$	8·13	21	$100\,F_3/F_4$	5·36
7	$100\,T_6/T_5\,(R)$	8·15	22	$100\,S_5/S_4\,(R)$	6·39
8	$100\,P_3/P_4\,(R)$	3·02	23	$100\,S_5/S_4\,(L)$	6·46
9	$100\,P_3/P_4\,(L)$	3·06	24	$100\,T_2/T_3\,(L)$	6·21
10	$100\,F_2/F_1\,(R)$	3·98	25	$100\,T_2/T_3\,(R)$	6·23
11	$100\,F_2/F_1\,(L)$	4·15	26	$100\,O_6/O_7\,(R)$	4·59
12	$100\,P_2/P_1\,(R)$	5·95	27	$100\,O_6/O_7\,(L)$	4·33
13	$100\,P_2/P_1\,(L)$	5·61	28	$100\,Mx_3/Mx_1\,(R)$	3·32
14	$100\,O_8/O_9\,(R)$	5·37	29	$100\,Mx_3/Mx_1\,(L)$	3·42
15	$100\,O_8/O_9\,(L)$	5·13			

We will now arrange our homologous indices in the order of intensity of their correlation coefficients, see Table V, p. 436.

These give an average correlation of homologous indices of ·7414, marking a high relationship due to homology after we have removed the lateral dominance. The sources of the differences in the intensity of homologous influence on the various bone indices is not easy to discover*. It is true that all the indices do not belong to the same class; in the second, third and eighth indices in Table V,

and that P_3 and P_4 are the greatest lengths measured. The index of one stands at the top of the homology order and variations of the maxillary indices are on the average small. The index P_3/P_4 stands at the bottom of the homology order. It may, however, be noted that its variation is the least in Table IV. The sphenoid is one of the earliest bones to be laid down, and its indices stand second and third in the homology order. There will obviously be discordances in the interpretation of Tables IV and V on this basis, but investigations might prove profitable when the authors have devoted more time to a study of what is known regarding the development of the skull.

* See however the preceding footnote.

S_4 = distance from krotaphion to krotaphion, and S_7 = distance between the lowest points right and left on the suture between the medial-pterygoid plate and the palate bone, are transverse lengths crossing the median sagittal plane; and again P_1 is the common arc between the two parietal bones. Hence these three indices differ from the remaining nine as not being composed of homologous lengths on opposite sides of the skull, but have the denominators in the pair of correlated indices the same. But this cannot be said to throw much light on their relative positions in the above table. In choosing the ratios of bone lengths some attempts were made to take them so that we should obtain a rough ratio of length to breadth, this accounts for indices like Mx_2/Mx_1, Ml_2/Ml_1, F_2/F_1, T_2/T_3, P_2/P_1, T_6/T_5, O_8/O_7 and O_8/O_9. But these eight indices show great differences in their correlations, and are in no way classed in position together. P_2, P_1, T_6, T_5, O_8, O_7, O_9 end chiefly at cranial points, but we are far from certain that that gives them a reduced homologous correlation, and as a matter of fact the lengths of the first six indices are in very much the same condition as to their terminals.

TABLE V. *Order of Correlation Coefficients for Homologous Indices.*

Order	Index	Correlation Coefficient	Order	Index	Correlation Coefficient
1	$100\,Mx_2/Mx_1$	$+\cdot9344\pm\cdot0038$	7	$100\,Mx_3/Mx_1$	$+\cdot7894\pm\cdot0125$
2	$100\,S_6/S_7$	$+\cdot9048\pm\cdot0043$	8	$100\,P_2/P_1$	$+\cdot7892\pm\cdot0093$
3	$100\,S_5/S_4$	$+\cdot8822\pm\cdot0056$	9	$100\,T_6/T_5$	$+\cdot6185\pm\cdot0145$
4	$100\,Ml_2/Ml_1$	$+\cdot8806\pm\cdot0057$	10	$100\,O_8/O_7$	$+\cdot6044\pm\cdot0146$
5	$100\,F_2/F_1$	$+\cdot8171\pm\cdot0075$	11	$100\,O_8/O_9$	$+\cdot4769\pm\cdot0179$
6	$100\,T_2/T_3$	$+\cdot7934\pm\cdot0086$	12	$100\,P_3/P_4$	$+\cdot4057\pm\cdot0207$

All, we think, that at present we are justified in concluding is, that homology plays a large part in determining the shape as well as the size of homologous bones, but that while this influence has very significant differences as between one bone and a second, we cannot so far state why homology should be much more active in the maxillary or sphenoid bones than in the parietal or occipital. Far more correlations of homologous indices must be worked out before light can be thrown on this point.

We have seen by the mere process of averaging that there is no dominance of either side of the skull in variation *as a whole*. But this does not exclude dominance in the case of individual bones. Table VI indicates that such dominance most probably exists in the case of two indices, although the ratio Δ_{R-L} to its probable error is not "markedly" significant.

In the case of both bones, where dominance of indicial variability is probably significant, that dominance is on the right side. But before we consider this point, it is worth while to inquire whether there is significance in the series of twelve ratios as a *whole*; for this purpose we apply the P_{λ_n} test. Multiplying the last column by ·67449, we shall have the ratio of the deviation to the standard deviation

TABLE VI. *Difference of Absolute Variability for Homologous Indices.*

Category of Significance of Difference	Index	Difference and its Probable Error $\Delta_{R-L} \pm$ p.e. of Δ_{R-L}	Ratio of Difference to its Probable Error Δ_{R-L}/p.e. of Δ_{R-L}
Probably significant	$100\,P_2/P_1$	+·3382±·0880	+3·84
	$100\,O_8/O_7$	+·2587±·0821	+3·15
Possibly, but far from definitely, significant	$100\,Ml_2/Ml_1$	−·1741±·0709	−2·46
	$100\,O_8/O_9$	+·2389±·1068	+2·24
	$100\,F_2/F_1$	−·1734±·0855	−2·03
Most probably insignificant	$100\,Mx_3/Mx_1$	−·0965±·0686	−1·41
	$100\,S_6/S_7$	−·1030±·0786	−1·31
	$100\,S_5/S_4$	−·0747±·0759	−0·98
	$100\,P_3/P_4$	−·0414±·0690	−0·60
	$100\,Mx_2/Mx_1$	−·0256±·0576	−0·44
	$100\,T_2/T_3$	+·0197±·0879	+0·22
	$100\,T_6/T_5$	+·0178±·1501	+0·12

of that deviation, and the probability integral table of the normal curve* provides us at once with the probability integrals of the twelve values. In Table VII the corresponding logarithms and their sums are provided.

TABLE VII.

Criterion for the General Significance of the Twelve Ratios in Table VI.

Order	Deviation / S.D. of Deviation	p	$\log p$	$p'=1-p$	$\log p'$
1	+2·590	·99520	$\bar{1}$·997,8231	·00480	$\bar{3}$·681,2412
2	+2·125	·98321	$\bar{1}$·992,6463	·01679	$\bar{2}$·225,0507
3	−1·659	·04856	$\bar{2}$·686,2787	·95144	$\bar{1}$·978,3814
4	+1·511	·93460	$\bar{1}$·970,6258	·06540	$\bar{2}$·815,5777
5	−1·369	·08550	$\bar{2}$·931,9661	·91450	$\bar{1}$·961,1837
6	−0·351	·17080	$\bar{1}$·232,4879	·82920	$\bar{1}$·918,6593
7	−0·884	·18835	$\bar{1}$·274,9656	·81165	$\bar{1}$·909,3688
8	−0·661	·25431	$\bar{1}$·405,3634	·74569	$\bar{1}$·872,5583
9	−0·405	·34274	$\bar{1}$·534,9648	·65726	$\bar{1}$·817,7372
10	−0·297	·38323	$\bar{1}$·583,4596	·61677	$\bar{1}$·790,1232
11	+0·148	·55883	$\bar{1}$·747,2797	·44117	$\bar{1}$·644,6060
12	+0·081	·53228	$\bar{1}$·726,1401	·46772	$\bar{1}$·669,9859
		$\log_{10}\lambda_n = -5{\cdot}915.9989$		$\log_{10}\lambda'_n = -6{\cdot}715,5266$	

$$\sqrt{n}\log_{10}e = \sqrt{12}\times{\cdot}434,2945 = 1{\cdot}504,4403.$$

$$u = -\log_{10}\lambda_n/(\sqrt{n}\log_{10}e) = 3{\cdot}9324, \quad u' = -\log_{10}\lambda'_n/(\sqrt{n}\log_{10}e) = 4{\cdot}4638.$$

* The true curve of distribution of the standard deviations of samples from a normal curve is given by Helmert's equation—a Type III curve, but in the case of a large sample this passes into a normal

Hence $Q_{\lambda n} = 1 - I(11, 3{\cdot}9324)$, $Q_{\lambda' n} = 1 - I(11, 4{\cdot}4638)$,

where $I(p, u)$ is the gamma integral ratio as tabled in the *Incomplete* Γ-*Function Tables*. Since $Q_{\lambda' n}$ is smaller than $Q_{\lambda n}$, it suffices to consider its value.

Thus we have $Q_{\lambda' n} = {\cdot}1562$,

and this does not indicate any great improbability of the twelve differences being due to random sampling from a population of no difference between R and L.

It is possible therefore when treating the twelve differences as a whole to consider them as random samples from populations having equal variation, notwithstanding that treated individually the first two differences might be considered as probably significant. This point—i.e. of the difference between a single Δ/(p.e. of Δ) and a whole series of such values—has too often been neglected in craniometric memoirs.

(5) *Comparison of the Influence of Homology on Index Correlations and Absolute Lengths.*

Returning to Table V, we recall the fact that the average correlation of the twelve homologous indices is ·7414. For the twenty-five absolute homologous lengths, measured on the separate cranial bones, it is ·8198. Thus on the average the absolute homologous lengths were more closely related than the indices, i.e. the sizes of the bones were more closely related than their shapes.

We may analyse this result a little further in the following table:

Correlation of Homologous Characters.

Bone	Mean Index Correlation	Mean Absolute Lengths Correlation
Sphenoid	·8935	·8206
Malar	·8806	·9309
Maxillary	·8619	·9198
Frontal	·8171	·9221
Temporal	·7059	·7894
Parietal	·5974	·7253
Occipital	·5407	·7087
All cases	·7414	·8198

It will be noticed that in both cases, index and absolute lengths, the temporal, parietal and occipital bones are the least highly correlated. The facial bones,

curve, with mean at the modal value of the S.D. But the difference of two quantities each following a normal distribution, if divided by the standard deviation of that difference, follows Student's z curve, which again passes into a normal curve for large samples. Since each pair of homologous bones has the same number of individuals, we may take a series of probability integrals, each corresponding to a normal curve based upon different-sized samples for each pair of different homologous indices.

malar, maxillary and frontal, form also, in both cases, a group with high correlations. The sphenoid stands at the top in the index column, and intermediate between the two groups in the absolute lengths column. With this one exception the influence of homology is more intense in the case of every bone in regard to size than to shape; also, apart from the sphenoid, the order of intensity of the influence of homology is much the same for indices and lengths. The greater symmetry of the face over the brain box is a noteworthy fact. Possibly it has been due to sexual selection; we notice failure of homology in the face far more than in the case of the retroceding calvaria.

(6) *Method of studying the Index Correlations of Individual Cranial Bones.*

In order to study these index correlations, we shall require to classify our cranial bones. Such bones may be either on the same or opposite sides of the median sagittal plane. If on the same side, they may be either contiguous or non-contiguous. If on opposite sides, they may be homologous or non-homologous. If non-homologous, they may be contiguous or non-contiguous with their homologous bone. The consideration of contiguity is *a priori* of great importance because it is not unreasonable to consider that contiguity would have considerable influence on measurements of *shape**.

In Table VIII we have classified the cranial bones with regard to their "contiguity." On the basis of these measures of contiguity, and taking into account that the components of some of the indices do not possess "laterality," i.e. are measured across the median sagittal plane or along it, we can divide up our correlations for purposes of study into the following classes:

(i) Correlation of Indices of Contiguous Bones (same side).

(ii) Correlation of Indices of slightly Contiguous Bones (same side).

(iii) Correlation of Indices of Non-contiguous Bones (same side).

(iv) Cross-Correlations. Index of a Bone with the Index of a Bone contiguous with the homologue of the first (on the opposite side).

(v) Cross-Correlations. Index of a Bone with the Index of a Bone non-contiguous with the homologue of the first (on the opposite side).

(vi) Correlation of two Indices of the same Bone.

(vii) Correlation of the same Indices on Homologous Bones†.

(viii) Correlation of different Indices on Homologous Bones.

* Where the union of two bones is an "overlap" (e.g. temporal and parietal bones) and not a serrated suture, the measurements have been taken to the edge of the overlap (e.g. squamous suture), the skulls not having been separated into their individual bones. This may have some influence on the correlation of the indices of contiguous bones, but it has not been possible to trace it. Nor have we been able to find in the index correlations differences associated with ossification in cartilage and development in membrane.

† This has already been discussed: see pp. 436—438 above.

TABLE VIII. *Contiguity of Cranial Bones.*

Bone	Frontal	Parietal	Temporal	Sphenoid	Occipital	Maxillary	Malar
Frontal	—	Contiguous	Non-contiguous	Contiguous	Non-contiguous	Slightly (*a*) contiguous	Contiguous
Parietal	Contiguous	—	Contiguous	Slightly (*b*) contiguous	Contiguous	Non-contiguous	Non-contiguous
Temporal	Non-contiguous	Contiguous	—	Contiguous	Contiguous	Non-contiguous	Slightly (*c*) contiguous
Sphenoid	Contiguous	Slightly (*b*) contiguous	Contiguous	—	Contiguous	Contiguous	Contiguous
Occipital	Non-contiguous	Contiguous	Contiguous	Contiguous	—	Non-contiguous	Non-contiguous
Maxillary	Slightly (*a*) contiguous	Non-contiguous	Non-contiguous	Contiguous	Non-contiguous	—	Contiguous
Malar	Contiguous	Non-contiguous	Slightly (*c*) contiguous	Contiguous	Non-contiguous	Contiguous	—

(*a*) Superior nasal point to Dacryon.
(*b*) From Sphenion to Krotaphion.
(*c*) On the temporal process.

There are many degrees of "Contiguity," but the above notes will best explain what we mean by "Slight Contiguity."

(ix) Correlation of Indices on Contiguous Bones, one Index of the pair having no laterality.

(x) Correlation of Indices on Non-contiguous Bones, one or both Indices of the pair having no laterality.

Under these classes we shall study our index correlations.

(7) *Correlation of Indices of Non-contiguous Bones* (*on same side*).

We will start with Group (iii) as most convenient for our present purposes.

Table IX indicates that we have 40 index correlations in our series of 29 indices—of which 27 have been worked out—when we confine our attention to non-contiguous bones of the same side. In this series those of highest value, as a rule positive, are between indices of the occipital and parietal bones and the maxillary index Mx_2/Mx_1. There seems little obvious reason why a breadth-height index of the maxillary should be less closely associated with a height-length index of the parietal, and in a *positive* sense, than with such indices as O_8/O_7 and O_8/O_9 of the distant parts of the occipital. Sometimes, as in the cases of F_2/F_1 with T_2/T_3 and O_8/O_9 with Ml_2/Ml_1, the right and left sides give correlations of opposite sign. Again, the correlations on the two sides, while differing within the limits of random sampling, differ fairly considerably from each other. Without being able to give any explanation of why certain correlations were higher than others, we note that all the correlations are less than 0·1, and thus perfectly useless for the purpose of predicting the shape of one bone from that of a second non-contiguous bone on the same side.

TABLE IX. *Correlation of Indices of Non-contiguous Bones (on same side).*

Index Pair	Correlation and p.e.	Index Pair	Correlation and p.e.
$O_8/O_9\,(R)$, $Mx_2/Mx_1\,(R)$	$+\cdot0674\pm\cdot0284$	$F_2/F_1\,(R)$, $T_2/T_3\,(R)$	$+\cdot0387\pm\cdot0230$
$O_8/O_9\,(L)$, $Mx_2/Mx_1\,(L)$	$+\cdot0819\pm\cdot0283$	$F_2/F_1\,(L)$, $T_2/T_3\,(L)$	$-\cdot0071\pm\cdot0230$
$P_3/P_4\,(R)$, $Mx_2/Mx_1\,(R)$	$+\cdot0554\pm\cdot0297$	$O_8/O_9\,(R)$, $Ml_2/Ml_1\,(R)$	$+\cdot0131\pm\cdot0249$
$P_3/P_4\,(L)$, $Mx_2/Mx_1\,(L)$	$+\cdot0782\pm\cdot0248$	$O_8/O_9\,(L)$, $Ml_2/Ml_1\,(L)$	$-\cdot0361\pm\cdot0247$
$O_8/O_7\,(R)$, $Mx_2/Mx_1\,(R)$	$+\cdot0744\pm\cdot0284$	$T_6/T_5\,(R)$, $Mx_3/Mx_1\,(R)$	$-\cdot0308\pm\cdot0311$
$O_8/O_7\,(L)$, $Mx_2/Mx_1\,(L)$	—	$T_6/T_5\,(L)$, $Mx_3/Mx_1\,(L)$	—
$T_2/T_3\,(R)$, $Mx_3/Mx_1\,(R)$	$+\cdot0711\pm\cdot0307$	$F_2/F_1\,(R)$, $T_6/T_5\,(R)$	$+\cdot0245\pm\cdot0231$
$T_2/T_3\,(L)$, $Mx_3/Mx_1\,(L)$	—	$F_2/F_1\,(L)$, $T_6/T_5\,(L)$	—
$F_2/F_1\,(R)$, $O_8/O_7\,(R)$	$+\cdot0677\pm\cdot0230$	$O_8/O_9\,(R)$, $Mx_3/Mx_1\,(R)$	$-\cdot0191\pm\cdot0310$
$F_2/F_1\,(L)$, $O_8/O_7\,(L)$	—	$O_8/O_9\,(L)$, $Mx_3/Mx_1\,(L)$	—
$P_3/P_4\,(R)$, $Ml_2/Ml_1\,(R)$	$-\cdot0406\pm\cdot0258$	$P_2/P_1\,(R)$, $Ml_2/Ml_1\,(R)$	$-\cdot0133\pm\cdot0258$
$P_3/P_4\,(L)$, $Ml_2/Ml_1\,(L)$	$-\cdot0652\pm\cdot0258$	$P_2/P_1\,(L)$, $Ml_2/Ml_1\,(L)$	—
$T_2/T_3\,(R)$, $Mx_2/Mx_1\,(R)$	$+\cdot0375\pm\cdot0282$	$P_2/P_1\,(R)$, $Mx_3/Mx_1\,(R)$	$+\cdot0092\pm\cdot0323$
$T_2/T_3\,(L)$, $Mx_2/Mx_1\,(L)$	$+\cdot0632\pm\cdot0282$	$P_2/P_1\,(L)$, $Mx_3/Mx_1\,(L)$	—
$P_2/P_1\,(R)$, $Mx_2/Mx_1\,(R)$	$-\cdot0535\pm\cdot0296$	$F_2/F_1\,(R)$, $O_8/O_9\,(R)$	$+\cdot0044\pm\cdot0232$
$P_2/P_1\,(L)$, $Mx_2/Mx_1\,(L)$	—	$F_2/F_1\,(L)$, $O_8/O_9\,(L)$	$+\cdot0073\pm\cdot0231$
$P_3/P_4\,(R)$, $Mx_3/Mx_1\,(R)$	$+\cdot0532\pm\cdot0322$	$O_8/O_7\,(R)$, $Mx_3/Mx_1\,(R)$	$-\cdot0046\pm\cdot0310$
$P_3/P_4\,(L)$, $Mx_3/Mx_1\,(L)$	—	$O_8/O_7\,(L)$, $Mx_3/Mx_1\,(L)$	—
$T_6/T_5\,(R)$, $Mx_2/Mx_1\,(R)$	$+\cdot0529\pm\cdot0285$	$O_8/O_7\,(R)$, $Ml_2/Ml_1\,(R)$	$+\cdot0019\pm\cdot0249$
$T_6/T_5\,(L)$, $Mx_2/Mx_1\,(L)$	—	$O_8/O_7\,(L)$, $Ml_2/Ml_1\,(L)$	—

As there were only 9 negative as against 18 positive correlations, we doubted whether the whole series could be treated as a random sample from a set of populations with zero correlation in each. Yet it seemed worth while examining this. We applied the $P_{\lambda n}$ test, treating samples (in all cases over 430) as having their r's distributed normally. We found that the more stringent set of the probability integrals gave

$$Q_{\lambda' n} = \cdot0066,$$

or, only in about 7 in 1000 trials would so poor a result arise in such large samples from populations having zero correlation between the indices*. Thus we feel bound to discard the hypothesis of these indices even in the case of non-contiguous bones being uncorrelated. Yet in the bulk of cases the indices are so small and irregular that we do not feel able to give much scope to a "general control factor" regulating the shapes of non-adjacent as well as adjacent bones.

If we leave out of account the correlations of O_8/O_9, O_8/O_7 and P_3/P_4 with Mx_2/Mx_1, then we find

$$Q_{\lambda' n} = \cdot1790,$$

or, it would not be an unreasonable assumption to suppose the remaining 22 index correlations to be really zero. This is a somewhat arbitrary procedure, yet it suffices to indicate how slender the correlations are. Indeed, if we applied a test

* Besides the hypothesis made in the text of zero r for very large samples being distributed normally, the r must be that of material which more or less closely follows a normal surface. If we adopt the assumption, not unreasonable, that the absolute measurements of the cranial bones do so, the indices will fail to do so with *theoretical* accuracy, but all the same with adequate *practical* sufficiency.

of thrice the probable error to each result individually, none of them would pass as certainly significant. Such a method of testing would not, however, be convincing when we have to deal, not with a single value, but with a large series of correlation coefficients.

Accordingly, to get a measure of the "general factor" controlling development of shape of the individual bones, we may find the mean correlation of the 27 non-contiguous bones = +·0197, and ask whether samples from 27 populations having this correlation, but their own standard deviations, could reasonably have given the 27 observed values; we find

$$Q_{\lambda' n} = \cdot 3452.$$

Accordingly, we may conclude that the general factor controlling the shapes of non-contiguous bones may amount to about +·02, while the factor of homology gives an average correlation of ·74, or has thirty-seven times the intensity of the former.

The startling conclusion here is the smallness of the relationship between the shapes of the skull at non-contiguous parts.

(8) *Correlation of Indices of slightly Contiguous Bones* (*on same side*). Group (ii).

The coefficients are given in Table X.

TABLE X. *Correlation of Indices of slightly Contiguous Bones* (*on same side*).

Index Pair	Correlation and p.e.	Index Pair	Correlation and p.e.
$P_3/P_4\,(R)$, $S_5/S_4\,(R)$	+·1205 ± ·0247	$F_2/F_1\,(R)$, $Mx_3/Mx_1\,(R)$	−·0409 ± ·0306
$P_3/P_4\,(L)$, $S_5/S_4\,(L)$	—	$F_3/F_1\,(L)$, $Mx_3/Mx_1\,(L)$	—
$P_2/P_1\,(R)$, $S_5/S_4\,(R)$	−·0674 ± ·0250	$F_2/F_1\,(R)$, $Mx_2/Mx_1\,(R)$	+·0407 ± ·0281
$P_2/P_1\,(L)$, $S_5/S_4\,(L)$	—	$F_3/F_1\,(L)$, $Mx_2/Mx_1\,(L)$	+·0093 ± ·0283
$T_6/T_5\,(R)$, $Ml_2/Ml_1\,(R)$	+·0596 ± ·0247	$P_3/P_4\,(R)$, $S_6/S_7\,(R)$	−·0359 ± ·0257
$T_6/T_5\,(L)$, $Ml_2/Ml_1\,(L)$	—	$P_3/P_4\,(L)$, $S_6/S_7\,(L)$	—
$P_2/P_1\,(R)$, $S_6/S_7\,(R)$	+·0482 ± ·0251	$T_2/T_3\,(R)$, $Ml_2/Ml_1\,(R)$	+·0113 ± ·0246
$P_2/P_1\,(L)$, $S_6/S_7\,(L)$	—	$T_2/T_3\,(L)$, $Ml_2/Ml_1\,(L)$	−·0004 ± ·0245

The only value which seems exceptional here is the correlation of the parietal P_3/P_4 with sphenoidal S_5/S_4, especially when we note the negative correlation of P_2/P_1 with the same sphenoidal index. Assuming that there was no arithmetical error in these values, the system of ten correlations was tested as a random sample from a zero population. Assuming normal distributions for the indices, their correlation coefficients r in the sampling would follow the curve $z = z_0\,(1 - r^2)^{\frac{1}{2}(n-4)}$. As no r is less than 434, and n can take values up to between 800 and 900, it is sufficient to use a normal distribution in each case for r. Calculating the probability integrals and applying the $P_{\lambda n}$ test, we found $Q_{\lambda' n} = \cdot 0086$. The ten values can scarcely therefore be looked upon as random samples from parent populations with

zero correlation. Omitting the first parietal-sphenoid correlation, we found for the nine remaining correlations, $Q_{\lambda' n} = \cdot 3266$, or the slightly contiguous group may in the great bulk be treated as without correlation.

The mean of the correlations of the indices for these slightly contiguous bones is $+\cdot 0145$. We next inquired whether these correlations could be considered as a random sample from ten populations having this value for their index correlations, but standard errors varying with the sizes of the samples. These sizes being large, we again supposed normal distribution of the coefficients and applied the $P_{\lambda n}$ test: we found $Q_{\lambda n} = \cdot 2312$ and $Q_{\lambda' n} = \cdot 1422$; the latter is the more stringent, but does not indicate that a random distribution of correlations from populations with a correlation of $\cdot 0145$ is very improbable.

It would seem, therefore, that the factor for "slight contiguity," giving a standard correlation of $\cdot 0145$, is not definitely greater than that for non-contiguity with a correlation of $\cdot 0197$. There appears, accordingly, no reason for separating the slightly contiguous from the non-contiguous bones, and we may say that the general factor linking the shapes of bones which are not to a considerable extent contiguous does not produce a general correlation exceeding $\cdot 02$.

(9) *Correlation of Indices of Contiguous Bones* (*on same side*). Group (i).

We now come to the study of our first group. From this group we shall be able to appreciate the maximum influence of contiguity on measurements of cranial shape. We thus reach Table XI, containing the index correlations of bones on the same side having fairly appreciable parts in contact. It will be seen at once that the condition of things is changed, 13 correlations out of 44 have numerically a value in excess of 0·1 and they range nearly up to $\cdot 16$. Their average is $-\cdot 01638$, or, say, $-\cdot 0164$*. It is difficult to lay much stress on negative as against positive *index* correlations, for the formation of the indices, after the choice of lengths, must be more or less arbitrary, and, as a rule, but not absolutely, the choice was directed to obtaining an index under 100. However, for the indices selected, the indices of contiguous bones give on the whole a negative correlation, or the effect of increasing contiguity is to reduce correlation ($\cdot 0197$, $\cdot 0145$, $-\cdot 0164$). If we consider the indices regardless of sign, then we find:

Non-contiguous bones	·0397
Slightly contiguous bones	·0434
Contiguous bones	·0655

These results confirm the view that the slightly contiguous may be classed with the non-contiguous, but indicate that contiguity gives an association somewhat less than double non-contiguity, if we neglect the sign of the correlations.

We will now test whether it is possible to consider our index correlations of contiguous bones as a random sample from 44 populations with the correlation

* The mean of the negative index correlations is $-\cdot 0783$, of the positive $+\cdot 0514^5$, and of the whole system without regard to sign ·06549.

TABLE XI. *Correlation of Cranial Indices on Contiguous Bones (on same side).*

Index Pair		Correlation and p.e.	Index Pair		Correlation and p.e.
P_3/P_4 (R),	O_8/O_7 (R)	$-\cdot1578\pm\cdot0238$	S_6/S_7 (L),	Mx_2/Mx_1 (L)	—
P_3/P_4 (L),	O_8/O_7 (L)	—	P_2/P_1 (R),	O_8/O_9 (R)	$-\cdot0540\pm\cdot0244$
Mx_2/Mx_1 (R),	Ml_2/Ml_1 (R)	$-\cdot1217\pm\cdot0290$	P_2/P_1 (L),	O_8/O_9 (L)	—
Mx_2/Mx_1 (L),	Ml_2/Ml_1 (L)	$-\cdot1472\pm\cdot0287$	S_6/S_7 (R),	O_8/O_9 (R)	$+\cdot0437\pm\cdot0241$
T_2/T_3 (R),	S_5/S_4 (R)	$+\cdot1167\pm\cdot0246$	S_6/S_7 (L),	O_8/O_9 (L)	—
T_2/T_3 (L),	S_5/S_4 (L)	$+\cdot1318\pm\cdot0239$	S_5/S_4 (R),	Ml_2/Ml_1 (R)	$+\cdot0435\pm\cdot0261$
P_2/P_1 (R),	O_8/O_7 (R)	$-\cdot1313\pm\cdot0240$	S_5/S_4 (L),	Ml_2/Ml_1 (L)	$+\cdot0011\pm\cdot0257$
P_2/P_1 (L),	O_8/O_7 (L)	—	T_6/T_5 (R),	O_8/O_7 (R)	$+\cdot0420\pm\cdot0234$
P_3/P_4 (R),	O_8/O_9 (R)	$-\cdot1237\pm\cdot0240$	T_6/T_5 (L),	O_8/O_7 (L)	—
P_3/P_4 (L),	O_8/O_9 (L)	$-\cdot1256\pm\cdot0243$	S_5/S_4 (R),	Mx_2/Mx_1 (R)	$+\cdot0396\pm\cdot0325$
Mx_3/Mx_1 (R),	Ml_2/Ml_1 (R)	$-\cdot1296\pm\cdot0313$	S_5/S_4 (L),	Mx_2/Mx_1 (L)	—
Mx_3/Mx_1 (L),	Ml_2/Ml_1 (L)	—	F_2/F_1 (R),	Ml_2/Ml_1 (R)	$+\cdot0368\pm\cdot0245$
P_3/P_4 (R),	T_2/T_3 (R)	$+\cdot1276\pm\cdot0240$	F_2/F_1 (L),	Ml_2/Ml_1 (L)	$+\cdot0146\pm\cdot0243$
P_3/P_4 (L),	T_2/T_3 (L)	$+\cdot1224\pm\cdot0240$	T_2/T_3 (R),	O_8/O_7 (R)	$+\cdot0350\pm\cdot0234$
P_2/P_1 (R),	T_2/T_3 (R)	$-\cdot1077\pm\cdot0241$	T_2/T_3 (L),	O_8/O_7 (L)	—
P_2/P_1 (L),	T_2/T_3 (L)	—	P_3/P_4 (R),	T_6/T_5 (R)	$-\cdot0346\pm\cdot0245$
F_2/F_1 (R),	S_5/S_4 (R)	$-\cdot1009\pm\cdot0242$	P_3/P_4 (L),	T_6/T_5 (L)	—
F_2/F_1 (L),	S_5/S_4 (L)	$-\cdot0981\pm\cdot0239$	S_5/S_4 (R),	O_8/O_9 (R)	$-\cdot0246\pm\cdot0248$
F_2/F_1 (R),	P_2/P_1 (R)	$-\cdot0992\pm\cdot0239$	S_5/S_4 (L),	O_8/O_9 (L)	$+\cdot0075\pm\cdot0245$
F_2/F_1 (L),	P_2/P_1 (L)	—	T_6/T_5 (R),	S_6/S_7 (R)	$+\cdot0205\pm\cdot0241$
T_6/T_5 (R),	O_8/O_9 (R)	$+\cdot0883\pm\cdot0234$	T_6/T_5 (L),	S_6/S_7 (L)	—
T_6/T_5 (L),	O_8/O_9 (L)	—	T_6/T_5 (R),	S_5/S_4 (R)	$-\cdot0160\pm\cdot0248$
S_5/S_4 (R),	Mx_2/Mx_1 (R)	$-\cdot0520\pm\cdot0301$	T_6/T_5 (L),	S_5/S_4 (L)	—
S_5/S_4 (L),	Mx_2/Mx_1 (L)	$-\cdot0692\pm\cdot0297$	T_2/T_3 (R),	S_6/S_7 (R)	$-\cdot0117\pm\cdot0240$
F_2/F_1 (R),	P_3/P_4 (R)	$+\cdot0689\pm\cdot0241$	T_2/T_3 (L),	S_6/S_7 (L)	—
F_2/F_1 (L),	P_3/P_4 (L)	$+\cdot0032\pm\cdot0243$	P_3/P_1 (R),	T_6/T_5 (R)	$+\cdot0070\pm\cdot0246$
S_6/S_7 (R),	Mx_3/Mx_1 (R)	$-\cdot0657\pm\cdot0309$	P_2/P_1 (L),	T_6/T_5 (L)	—
S_6/S_7 (L),	Mx_3/Mx_1 (L)	—	S_6/S_7 (R),	Ml_2/Ml_1 (R)	$-\cdot0057\pm\cdot0250$
T_2/T_3 (R),	O_8/O_9 (R)	$+\cdot0654\pm\cdot0234$	S_6/S_7 (L),	Ml_2/Ml_1 (L)	—
T_2/T_3 (L),	O_8/O_9 (L)	$+\cdot0604\pm\cdot0233$	F_2/F_1 (R),	S_6/S_7 (R)	$+\cdot0044\pm\cdot0238$
S_5/S_4 (R),	O_8/O_7 (R)	$-\cdot0620\pm\cdot0247$	F_2/F_1 (L),	S_6/S_7 (L)	—
S_5/S_4 (L),	O_8/O_7 (L)	—	S_6/S_7 (R),	O_8/O_7 (R)	$-\cdot0012\pm\cdot0241$
S_6/S_7 (R),	Mx_2/Mx_1 (R)	$-\cdot0617\pm\cdot0287$	S_6/S_7 (L),	O_8/O_7 (L)	—

$-\cdot0164$. Working this out by the $P_{\lambda n}$ test, we find $Q_{\lambda n}=\cdot000{,}0012$, or it is not possible to consider that we are dealing with random deviations from an average relationship measured by $-\cdot0164$. This is really clear from Table XI itself, for it would appear that the correlation between certain bones is much higher, significantly higher, than that between others.

To see whether it is feasible to throw any light on the matter we have drawn up Table XII, which gives the average correlation between the indices of bones on the same side, whether adjacent or not. First values are those regardless of sign, the second values in square brackets pay attention to sign.

Confining our attention here to the mean correlation values independent of sign we see that the shape of the parietal has the closest average relationship to the shapes of other cranial bones. The highest average correlation with an individual bone is that with the occipital, where all the correlations are negative; this might

TABLE XII. *Average Relationship of Bone Indices (on same side) having regard to Contiguity*.*

Bone	Frontal	Parietal	Temporal	Sphenoid	Occipital	Maxillary	Malar
Frontal	—	Contiguous ·0571 [− ·0090]	Non-contiguous ·0234 [+ ·0187]	Contiguous ·0678 [− ·0649]	Non-contiguous ·0265 [+ ·0265]	Slightly contiguous ·0303 [+ ·0030]	Contiguous ·0257 [+ ·0257]
Parietal	Contiguous ·0571 [− ·0090]	—	Contiguous ·0799 [+ ·0229]	Slightly contiguous ·0680 [+ ·0163^5]	Contiguous ·1185 [− ·1185]	Non-contiguous ·0499 [+ ·0285]	Non-contiguous ·0397 [− ·0397]
Temporal	Non-contiguous ·0234 [+ ·0187]	Contiguous ·0799 [+ ·0229]	—	Contiguous ·0593 [+ ·0483]	Contiguous ·0582 [+ ·0582]	Non-contiguous ·0511 [+ ·0388]	Slightly contiguous ·0238 [+ ·0235]
Sphenoid	Contiguous ·0678 [− ·0649]	Slightly contiguous ·0680 [+ ·0163^5]	Contiguous ·0593 [+ ·0483]	—	Contiguous ·0278 [− ·0073]	Contiguous ·0576 [− ·0418]	Contiguous ·0168 [+ ·0130]
Occipital	Non-contiguous ·0265 [+ ·0265]	Contiguous ·1185 [− ·1185]	Contiguous ·0582 [+ ·0582]	Contiguous ·0278 [− ·0073]	—	Non-contiguous ·0495 [+ ·0400]	Non-contiguous ·0170 [− ·0070]
Maxillary	Slightly contiguous ·0303 [+ ·0030]	Non-contiguous ·0499 [+ ·0285]	Non-contiguous ·0511 [+ ·0388]	Contiguous ·0576 [− ·0418]	Non-contiguous ·0495 [+ ·0400]	—	Contiguous ·1328 [− ·1328]
Malar	Contiguous ·0257 [+ ·0257]	Non-contiguous ·0397 [− ·0397]	Slightly contiguous ·0238 [+ ·0235]	Contiguous ·0168 [+ ·0130]	Non-contiguous ·0170 [− ·0070]	Contiguous ·1328 [− ·1328]	—
Total	·0393 [− ·0015]	·0722 [− ·0166]	·0553 [+ ·0377]	·0500 [− ·0038]	·0539 [− ·0031]	·0588 [− ·0024]	·0436 [− ·0222]

* Based on Tables IX, X and XI.

be attributed to the length of contour in contact, and such a view might be considered as confirmed by the second highest parietal correlation being with the temporal, but it is hardly in keeping with its relationship to the only slightly contiguous sphenoid or with the far more contiguous frontal. Indeed, while extent of contiguity plays a part in these correlations, it is clearly not the only factor to be considered. Thus the highest individual average correlation is that between malar and maxillary, where all the separate correlations are negative, yet the mere extent of contact between the latter pair is hardly in excess of that between parietal and occipital, or between parietal and frontal. Clearly the correlation will depend not only on the extent of contours in contact, but also on the extent relative to the size of the bones in contact. Indeed there are many other factors which may take part in determining these correlations—not only the original nature of the bones themselves, but the order in which they are laid down, etc. What is the really important point to be regarded is this: that, while many of the correlations are insignificant, some are certainly significant, but they are in the latter case so small that any idea of predicting the shape of one bone from the shape of an adjacent bone is idle*. As illustration let us take the maxillary index Mx_2/Mx_1 and the malar index Ml_2/Ml_1 on the right side of the skull; with the relatively high correlation $-\cdot1217 \pm \cdot0290$, the prediction formula of the latter from the former is by the values in Tables IV and XI:

$$\text{Mean}\left(\frac{Ml_2}{Ml_1}\right) \text{ for array} = \text{Mean}\left(\frac{Ml_2}{Ml_1}\right) \text{ for population} - \cdot1217 \times \frac{5\cdot8490}{5\cdot3937}\left(\frac{Mx_2}{Mx_1} - 83\cdot1522\right)$$

$$= \quad " \quad " \quad - \cdot1320\left(\frac{Mx_2}{Mx_1} - 83\cdot1522\right).$$

In other words, if the maxillary index be raised by a unit above its mean value, the malar index would be lowered only by ·1320 of a unit, while the standard error of a determination from the above formula is 5·8055, or is only some ·045 less than the standard deviation of the entire population!

Our general conclusion must be that contiguity does influence to a small extent the indices of shape of the cranial bones. But the influence is so small and the factors determining it so little at present investigated, that very little error would be introduced by supposing for the time being that all these indices whether of contiguous or non-contiguous bones are independent.

The striking feature of the investigation up to this point is the intensity of the homology factor as compared with the adjacency factor.

If, notwithstanding these general considerations, the reader desires a reasonable scheme to represent the whole distribution of contiguous bone indices, we may

* The reader must bear in mind a certain apparent paradox: Two adjacent bones will normally always "fit," and it might be considered that this indicated a high correlation of shapes. But this is not so, the shapes of the two bones follow no general law. Thus, if we take the correlation of the parietal and occipital indices, P_2/P_1 and O_3/O_2, $= -\cdot0540 \pm \cdot0244$, which cannot be considered individually as certainly significant, the mean of the group of occipital indices corresponding to a given parietal index will scarcely differ at all from the mean occipital index value; thus there is no sensible association of one index with a second.

consider whether treating all the correlations as of one sign, but with their individual standard errors, they could be reasonably represented by a random sample from a population with correlation equal to the average numerical value ·0655. The computing labour is somewhat lengthy, and since the standard errors are nearly equal might reasonably have been lightened by treating them as the same, and equal to their mean, but to avoid a not unusual type of criticism, they have been kept distinct. The numbers on which the correlations are based (see Tables I and III) are very considerable, and the correlations themselves are small and many insignificant; we have thus no hesitation in treating their distribution as normal. The P_{λ_n} test being used, the probability integrals and their logarithms were obtained for the two opposite directions, with the following results:

Probability Integrals: From left to right	From right to left
$\log_{10}\lambda_n = -22{\cdot}683{,}6667,$	$\log_{10}\lambda'_n = -22{\cdot}639{,}2892,$
$\sqrt{n}\log_e 10 = \sqrt{44}\log_e 10 = 2{\cdot}880{,}7838,$	$= 2{\cdot}880{,}7838,$
$u = 7{\cdot}87413,$	$u' = 7{\cdot}85873,$
$P_{\lambda_n} = I(43,\ 7{\cdot}87413),$	$P_{\lambda'_n} = I(43,\ 7{\cdot}85873).$

Since n is greater than 30, Miss F. N. David's Table* cannot be used, and resort must be had to the *Tables of the Incomplete* Γ-*Function*†. Hence we find:

$P_{\lambda_n} = {\cdot}8888,$	$P_{\lambda'_n} = {\cdot}8863,$
$Q_{\lambda_n} = {\cdot}1112,$	$Q_{\lambda'_n} = {\cdot}1127.$

There is scarcely any difference between the two sets, or if the parent populations consisted of a series of surfaces of correlation $\rho = {\cdot}0655$ we should obtain, once in nine random samples, a sample more improbable than the observed one. This does not justify us in asserting that the 44 index correlations of contiguous bones were so obtained, but it does justify us in stating that a moderately reasonable graduation of the observed results may be summed up in random sampling from a parent population of correlation coefficients with mean $\rho = {\cdot}0655$. And this gives far greater definition to the system than the simple statement that the mean of the correlations is ·0655.

(10) *Cross-Correlations.* Groups (iv) and (v).

Some remarkable results flow from these cross-correlations, indicating how great is the influence of homology and how small that of contiguity. We divide our indices into two classes, namely:

A. The correlation of an index of one bone with the index of a second bone on the opposite side contiguous with the homologue of the first bone. The values of such correlations are given in Table XIII, p. 448. Out of 66 possible values, owing to the great labour already spent in making correlation tables and working them out (see p. 433), only 22 values were determined.

* *Biometrika*, Vol. xxvi. 1934, pp. 1—11. Miss David gives P_{λ_n} for $\log_{10}\lambda_n$.

† Reissue *Biometrika* Office, University College, London, price 42*s.* net.

TABLE XIII.

Cross-Correlations. A. *Index of a Bone with the Index of a Second Bone on the other side contiguous with the homologue of the First*.*

Index Pair	Correlation and p.e.	Index Pair	Correlation and p.e.
$T_2/T_3(R)$, $S_5/S_4(L)$	$+\cdot1208\pm\cdot0240$	$P_2/P_1(L)$, $T_2/T_3(R)$	—
$T_2/T_3(L)$, $S_5/S_4(R)$	$+\cdot1139\pm\cdot0243$	$P_2/P_1(R)$, $O_8/O_9(L)$	—
$Mx_2/Mx_1(R)$, $Ml_2/Ml_1(L)$	$-\cdot0905\pm\cdot0290$	$P_2/P_1(L)$, $O_8/O_9(R)$	—
$Mx_2/Mx_1(L)$, $Ml_2/Ml_1(R)$	$-\cdot1185\pm\cdot0293$	$P_2/P_1(R)$, $O_8/O_7(L)$	—
$P_3/P_4(R)$, $O_8/O_9(L)$	$-\cdot1142\pm\cdot0242$	$P_2/P_1(L)$, $O_8/O_7(R)$	—
$P_3/P_4(L)$, $O_8/O_9(R)$	$-\cdot0809\pm\cdot0245$	$P_3/P_4(R)$, $O_8/O_7(L)$	—
$P_3/P_4(R)$, $T_2/T_3(L)$	$+\cdot0784\pm\cdot0243$	$P_3/P_4(L)$, $O_8/O_7(R)$	—
$P_3/P_4(L)$, $T_2/T_3(R)$	$+\cdot0999\pm\cdot0245$	$T_2/T_3(R)$, $O_8/O_7(L)$	—
$T_2/T_3(R)$, $O_8/O_9(L)$	$+\cdot0823\pm\cdot0233$	$T_2/T_3(L)$, $O_8/O_7(R)$	—
$T_2/T_3(L)$, $O_8/O_9(R)$	$+\cdot0757\pm\cdot0233$	$T_6/T_5(R)$, $O_8/O_7(L)$	—
$F_2/F_1(R)$, $S_5/S_4(L)$	$-\cdot0700\pm\cdot0240$	$T_6/T_5(L)$, $O_8/O_7(R)$	—
$F_2/F_1(L)$, $S_5/S_4(R)$	$-\cdot0695\pm\cdot0243$	$T_6/T_5(R)$, $O_8/O_9(L)$	—
$S_5/S_4(R)$, $Mx_2/Mx_1(L)$	$-\cdot0609\pm\cdot0301$	$T_6/T_5(L)$, $O_8/O_9(R)$	—
$S_5/S_4(L)$, $Mx_2/Mx_1(R)$	—	$T_2/T_3(R)$, $S_6/S_7(L)$	—
$F_2/F_1(R)$, $P_2/P_1(L)$	$+\cdot0549\pm\cdot0242$	$T_2/T_3(L)$, $S_6/S_7(R)$	—
$F_2/F_1(L)$, $P_2/P_1(R)$	—	$T_6/T_7(R)$, $S_6/S_7(L)$	—
$S_6/S_7(R)$, $Mx_3/Mx_1(L)$	—	$T_6/T_7(L)$, $S_6/S_7(R)$	—
$S_6/S_7(L)$, $Mx_3/Mx_1(R)$	$-\cdot0427\pm\cdot0298$	$T_6/T_5(R)$, $S_5/S_4(L)$	—
$S_5/S_4(R)$, $Ml_2/Ml_1(L)$	$+\cdot0211\pm\cdot0260$	$T_6/T_5(L)$, $S_5/S_4(R)$	—
$S_5/S_4(L)$, $Ml_2/Ml_1(R)$	$+\cdot0425\pm\cdot0258$	$S_6/S_7(R)$, $O_8/O_9(L)$	—
$S_5/S_4(R)$, $O_8/O_9(L)$	$-\cdot0276\pm\cdot0248$	$S_6/S_7(L)$, $O_8/O_9(R)$	—
$S_5/S_4(L)$, $O_8/O_9(R)$	$-\cdot0075\pm\cdot0245$	$S_6/S_7(R)$, $O_8/O_7(L)$	—
$F_2/F_1(R)$, $P_3/P_4(L)$	$+\cdot0267\pm\cdot0242$	$S_6/S_7(L)$, $O_8/O_7(R)$	—
$F_2/F_1(L)$, $P_3/P_4(R)$	—	$S_5/S_4(R)$, $O_8/O_7(L)$	—
$F_2/F_1(R)$, $Ml_2/Ml_1(L)$	$+\cdot0205\pm\cdot0243$	$S_5/S_4(L)$, $O_8/O_7(R)$	—
$F_2/F_1(L)$, $Ml_2/Ml_1(R)$	$+\cdot0235\pm\cdot0244$	$S_5/S_4(R)$, $Mx_3/Mx_1(L)$	—
$F_2/F_1(R)$, $S_6/S_7(L)$	—	$S_5/S_4(L)$, $Mx_3/Mx_1(R)$	—
$F_2/F_1(L)$, $S_6/S_7(R)$	—	$S_6/S_7(R)$, $Ml_2/Ml_1(L)$	—
$P_2/P_1(R)$, $T_6/T_7(L)$	—	$S_6/S_7(L)$, $Ml_2/Ml_1(R)$	—
$P_2/P_1(L)$, $T_6/T_7(R)$	—	$S_6/S_7(R)$, $Mx_2/Mx_1(L)$	—
$P_3/P_4(R)$, $T_6/T_5(L)$	—	$S_6/S_7(L)$, $Mx_2/Mx_1(R)$	—
$P_3/P_4(L)$, $T_6/T_5(R)$	—	$Mx_3/Mx_1(R)$, $Ml_2/Ml_1(L)$	—
$P_2/P_1(R)$, $T_2/T_3(L)$	—	$Mx_3/Mx_1(L)$, $Ml_2/Ml_1(R)$	—

B. The correlation of an index on one bone with the index of a bone on the opposite side non-contiguous with the homologue of the first bone. The values of such correlations are given in Table XIV. In the same table are given the slightly contiguous values. In the first case there are 14 out of 40 correlations, and in the second case only 6 out of 16. If these be pooled there will be 20 out of 56. For the purpose of comparison we give some results in Table XV below.

These results are noteworthy, they indicate that for the purely numerical values, which are really the more important ones (see p. 443), the cross-correlations do not fall short of the direct correlations either for contiguous or non-contiguous pairs;

* As in previous tables the correlations which have not been worked out are indicated. Had time and energy sufficed for working out the whole series, we do not believe that the results would have differed much from those of our practically random sample.

TABLE XIV.

Cross-Correlations. B. *Index of a Bone with the Index of a Second Bone on the other side non-contiguous or only slightly contiguous with the homologue of the First.*

Index Pair		Correlation and p.e.	Index Pair		Correlation and p.e.
$O_8/O_9\,(R)$,	$Mx_2/Mx_1\,(L)$	$+\cdot0910\pm\cdot0285$	$T_6/T_5\,(L)$,	$Mx_2/Mx_1\,(R)$	—
$O_8/O_9\,(L)$,	$Mx_2/Mx_1\,(R)$	$+\cdot0900\pm\cdot0283$	$T_6/T_5\,(R)$,	$Mx_3/Mx_1\,(L)$	—
$P_3/P_4\,(R)$,	$Mx_2/Mx_1\,(L)$	$+\cdot0669\pm\cdot0298$	$T_6/T_5\,(L)$,	$Mx_3/Mx_1\,(R)$	—
$P_3/P_4\,(L)$,	$Mx_2/Mx_1\,(R)$	$+\cdot0801\pm\cdot0308$	$O_8/O_9\,(R)$,	$Mx_3/Mx_1\,(L)$	—
$P_3/P_4\,(R)$,	$Ml_2/Ml_1\,(L)$	$-\cdot0246\pm\cdot0258$	$O_8/O_9\,(L)$,	$Mx_3/Mx_1\,(R)$	—
$P_3/P_4\,(L)$,	$Ml_2/Ml_1\,(R)$	$-\cdot0635\pm\cdot0261$	$O_8/O_7\,(R)$,	$Mx_2/Mx_1\,(L)$	—
$T_2/T_3\,(R)$,	$Mx_2/Mx_1\,(L)$	$+\cdot0593\pm\cdot0284$	$O_8/O_7\,(L)$,	$Mx_2/Mx_1\,(R)$	—
$T_2/T_3\,(L)$,	$Mx_2/Mx_1\,(R)$	$+\cdot0544\pm\cdot0283$	$O_8/O_7\,(R)$,	$Mx_3/Mx_1\,(L)$	—
$F_2/F_1\,(R)$,	$T_2/T_3\,(L)$	$-\cdot0515\pm\cdot0229$	$O_8/O_7\,(L)$,	$Mx_3/Mx_1\,(R)$	—
$F_2/F_1\,(L)$,	$T_2/T_3\,(R)$	$-\cdot0447\pm\cdot0231$	$O_8/O_7\,(R)$,	$Ml_2/Ml_1\,(L)$	—
$F_2/F_1\,(R)$,	$O_8/O_9\,(L)$	$+\cdot0157\pm\cdot0231$	$O_8/O_7\,(L)$,	$Ml_2/Ml_1\,(R)$	—
$F_2/F_1\,(L)$,	$O_8/O_9\,(R)$	$+\cdot0415\pm\cdot0232$			
$O_8/O_9\,(R)$,	$Ml_2/Ml_1\,(L)$	$-\cdot0076\pm\cdot0248$		Slightly contiguous	
$O_8/O_9\,(L)$,	$Ml_2/Ml_1\,(R)$	$-\cdot0382\pm\cdot0248$	$P_3/P_4\,(R)$,	$S_5/S_4\,(L)$	$+\cdot0722\pm\cdot0250$
$F_2/F_1\,(R)$,	$T_6/T_5\,(L)$	—	$P_3/P_4\,(L)$,	$S_5/S_4\,(R)$	$+\cdot0631\pm\cdot0256$
$F_2/F_1\,(L)$,	$T_6/T_5\,(R)$	—	$T_2/T_3\,(R)$,	$Ml_2/Ml_1\,(L)$	$+\cdot0063\pm\cdot0246$
$F_2/F_1\,(R)$,	$O_8/O_7\,(L)$	—	$T_2/T_3\,(L)$,	$Ml_2/Ml_1\,(R)$	$+\cdot0328\pm\cdot0246$
$F_2/F_1\,(L)$,	$O_8/O_7\,(R)$	—	$F_2/F_1\,(R)$,	$Mx_2/Mx_1\,(L)$	$+\cdot0196\pm\cdot0282$
$P_2/P_1\,(R)$,	$Mx_2/Mx_1\,(L)$	—	$F_2/F_1\,(L)$,	$Mx_2/Mx_1\,(R)$	$+\cdot0323\pm\cdot0287$
$P_2/P_1\,(L)$,	$Mx_2/Mx_1\,(R)$	—	$F_2/F_1\,(R)$,	$Mx_3/Mx_1\,(L)$	—
$P_2/P_1\,(R)$,	$Mx_3/Mx_1\,(L)$	—	$F_2/F_1\,(L)$,	$Mx_3/Mx_1\,(R)$	—
$P_2/P_1\,(L)$,	$Mx_3/Mx_1\,(R)$	—	$P_2/P_1\,(R)$,	$S_6/S_7\,(L)$	—
$P_2/P_1\,(R)$,	$Ml_2/Ml_1\,(L)$	—	$P_2/P_1\,(L)$,	$S_6/S_7\,(R)$	—
$P_2/P_1\,(L)$,	$Ml_2/Ml_1\,(R)$	—	$P_3/P_4\,(R)$,	$S_6/S_7\,(L)$	—
$P_3/P_4\,(R)$,	$Mx_3/Mx_1\,(L)$	—	$P_3/P_4\,(L)$,	$S_6/S_7\,(R)$	—
$P_3/P_4\,(L)$,	$Mx_3/Mx_1\,(R)$	—	$P_2/P_1\,(R)$,	$S_5/S_4\,(L)$	—
$T_2/T_3\,(R)$,	$Mx_3/Mx_1\,(L)$	—	$P_2/P_1\,(L)$,	$S_5/S_4\,(R)$	—
$T_2/T_3\,(L)$,	$Mx_3/Mx_1\,(R)$	—	$T_6/T_5\,(R)$,	$Ml_2/Ml_1\,(L)$	—
$T_6/T_5\,(R)$,	$Mx_2/Mx_1\,(L)$	—	$T_6/T_5\,(L)$,	$Ml_2/Ml_1\,(R)$	—

TABLE XV. *Means of Direct and Cross-Correlations of Indices.*

	Direct Correlations Means		Cross-Correlations Means	
	Arithmetical	Algebraical	Arithmetical	Algebraical
Contiguous	·0655	−·0164	·0656	+·0035
Non-contiguous	·0397	+·0197	·0478*	+·0248†

the only real difference occurs in the algebraical values of the contiguous pairs; even here the average algebraic correlation of the cross pairs is higher than for the direct pairs. The correlations are all small but many of them must be considered as

* This is ·0521 without the "slightly contiguous."

† +·0192 without the "slightly contiguous."

significant. The result is a striking instance of the power of homology, the correlation of the index of a bone with that of a second bone on its own side is on the average not more intense than it is with the homologue of the second bone on the opposite side. A comparison of the individual correlations in Table XIII with the corresponding correlations in Table XI shows to what extent this is true for individual pairs of contiguous bones, and a like comparison of Tables IX and XIV indicates the character of the relationship for non-contiguous individual bones. The reader will find some discrepancies where the difference is very considerable compared to the probable error of the difference. Thus compare

$$F_2/F_1(R), \quad P_2/P_1(R) = -\cdot0992 \text{ with } F_2/F_1(R), \quad P_2/P_1(L) = +\cdot0549.$$

Further compare

$$T_2/T_3(R), \quad O_8/O_9(R) = +\cdot0654 \text{ and } T_2/T_3(L), \quad O_8/O_9(L) = +\cdot0604$$
$$\text{with } T_2/T_3(L), \quad O_8/O_9(R) = +\cdot0757 \text{ and } T_2/T_3(R), \quad O_8/O_9(L) = +\cdot0823,$$

both cross-correlations being greater than both direct correlations. Again we may compare

$$F_2/F_1(R), \quad P_3/P_4(R) = +\cdot0689 \text{ and } F_2/F_1(L), \quad P_3/P_4(L) = +\cdot0032$$
$$\text{with } F_2/F_1(R), \quad P_3/P_4(L) = +\cdot0267;$$

and once more

$$F_2/F_1(R), \quad Ml_2/Ml_1(R) = +\cdot0368 \text{ and } F_2/F_1(L), \quad Ml_2/Ml_1(L) = +\cdot0146$$
$$\text{with } F_2/F_1(R), \quad Ml_2/Ml_1(L) = +\cdot0205 \text{ and } F_2/F_1(L), \quad Ml_2/Ml_1(R) = +\cdot0235.$$

Many of these results can be attributed to the random sampling, for the correlations are so small as to be of the same order as their probable errors. While it is possible that if the correlations had been worked out for the same pairs of bones in each case, the bones on the same side contiguous and non-contiguous might have shown on the average higher correlations than with the corresponding bones on the opposite side, the differences if significant are so small that we cannot assert that they exist when a random selection of indices has been worked out for each category and we base our conclusions on these. We can, however, assert that the factor of homology is so influential that with the small correlations which exist between the indices of non-homologous bones, we cannot prove on our data that the correlation of the index of a bone A on one side with the index of a bone B, contiguous or non-contiguous, on the same side is definitely greater than the correlation of the index of A with the index of the bone B' on the opposite side, where B' is the homologue of B.

(11) *Correlation of Indices with Laterality and without Laterality.*

We define an index with laterality as one based on the ratio of two lengths both taken on the same bone, that bone being situated on the right or left side of the skull. An index without laterality is based on the ratio of two lengths neither of which is peculiar to one side of the skull, but both are measured on the same bone. A list of these non-bilateral lengths is given on p. 427. We have confined our

attention to five such non-lateral indices, namely two on the frontal, F_3/F_4, F_5/F_4, and three on the occipital, O_4/O_1, O_6/O_3 and $O_5/(O_2+O_3)$. We will deal first with the non-contiguous, or only slightly contiguous values, our Group (x). The correlations are given in Table XVI. The mean value independent of sign for the non-contiguous correlations is ·0672, and for the slightly contiguous group ·0467 we have no hesitation in pooling the two series, for we should anticipate that the latter would have been the higher. The two series combined give a mean numerical value of ·0604. This is somewhat greater than the mean numerical value of the correlations for the contiguous bones in Table XVII (p. 452), i.e. ·0538. No clearer confirmation could be found of the statement that contiguity has little to do with the mean

TABLE XVI.

Correlation of Indices on non-contiguous Bones, one Index of the pair having no Laterality.

Index Pair		Correlation and p.e.	Index Pair		Correlation and p.e.
F_5/F_4,	$T_2/T_3(R)$	$+\cdot0893\pm\cdot0229$	F_5/F_4,	$O_8/O_9(L)$	—
F_5/F_4,	$T_2/T_3(L)$	—	F_3/F_4,	$O_8/O_9(R)$	$+\cdot0591\pm\cdot0231$
$O_5/(O_2+O_3)$,	$Mx_2/Mx_1(R)$	$+\cdot0754\pm\cdot0285$	F_3/F_4,	$O_8/O_9(L)$	—
$O_5/(O_2+O_3)$,	$Mx_2/Mx_1(L)$	—		Slightly contiguous	
F_5/F_4,	$T_6/T_5(R)$	$+\cdot0644\pm\cdot0231$	F_3/F_4,	$Mx_2/Mx_1(R)$	$+\cdot0571\pm\cdot0281$
F_5/F_4,	$T_6/T_5(L)$	—	F_3/F_4,	$Mx_2/Mx_1(L)$	—
F_3/F_4,	$T_6/T_5(R)$	$+\cdot0635\pm\cdot0231$	F_3/F_4,	$Mx_3/Mx_1(R)$	$-\cdot0526\pm\cdot0305$
F_3/F_4,	$T_6/T_5(L)$	—	F_3/F_4,	$Mx_3/Mx_1(L)$	—
F_3/F_4,	$O_8/O_7(R)$	$+\cdot0629\pm\cdot0230$	F_5/F_4,	$Mx_3/Mx_1(R)$	$-\cdot0401\pm\cdot0306$
F_3/F_4,	$O_8/O_7(L)$	—	F_5/F_4,	$Mx_3/Mx_1(L)$	—
O_4/O_1,	$Mx_3/Mx_1(R)$	$-\cdot0619\pm\cdot0311$	F_5/F_4,	$Mx_2/Mx_1(R)$	$+\cdot0368\pm\cdot0281$
O_4/O_1,	$Mx_3/Mx_1(L)$	—	F_5/F_4,	$Mx_2/Mx_1(L)$	—
F_5/F_4,	$O_8/O_9(R)$	$+\cdot0612\pm\cdot0231$			

values found for the series in Tables XIII and XIV. We have to admit that our correlations are far from an exhaustive list of possible correlations, which might be worked out, but those chosen have not been in any way selected to give high or low values of the correlations. If there be really any sensible difference between the non-contiguous pairs in Table XVI, and the contiguous pairs in Table XVII, we can hardly discover it on the values worked out thus far.

(12) *Correlation of Indices on Contiguous Bones, one of the Indices correlated having no Laterality.*

Of the 28 indices falling into Table XVII, no less than 19 are negative. In this the table differs markedly from Table XVI where out of 12 correlations 9 are positive. The negative correlations have a mean of $-\cdot0541$ and the positive a mean of $+\cdot0533$. When we consider only the numerical value the mean correlation is, as we have stated in the last section, ·0538. It is difficult however to lay stress on the near equality of the mean positive and negative correlations for we can change the sign

by inverting the ratio. Thus if we had taken for our index 100 O_1/O_4, which would have made the mean of that index under, instead of over, 100, we should have found $P_3/P_4(R)$ increasing with O_1/O_4, instead of decreasing as it does with O_4/O_1; similarly if we had taken our index 100 $(O_2+O_3)/O_5$, which would again have given a mean value under instead of over 100, we should have got the opposite sign for the index correlation $S_5/S_4(R)$, $(O_2+O_3)/O_5$ to that of $S_5/S_4(R)$ and $O_5/(O_2+O_3)$. In other words the signs of index correlations, as we have before stated, are at our choice, depending on which constituent of the ratio we assign to the numerator, which to the denominator.

TABLE XVII.

Correlation of Indices on Contiguous Bones, one of the Indices correlated having no Laterality.

Index Pair	Correlation and p.e.	Index Pair	Correlation and p.e.
$P_3/P_4(R)$, O_4/O_1	$-\cdot1296\pm\cdot0241$	F_5/F_4, $Ml_2/Ml_1(R)$	$-\cdot0607\pm\cdot0243$
$P_3/P_4(L)$, O_4/O_1	—	F_5/F_4, $Ml_2/Ml_1(L)$	—
$S_5/S_4(R)$, $O_5/(O_2+O_3)$	$-\cdot1154\pm\cdot0245$	F_3/F_4, $S_6/S_7(R)$	$+\cdot0439\pm\cdot0237$
$S_5/S_4(L)$, $O_5/(O_2+O_3)$	—	F_3/F_4, $S_6/S_7(L)$	—
F_5/F_4, $P_3/P_4(R)$	$+\cdot1103\pm\cdot0239$	$S_6/S_7(R)$, O_4/O_1	$-\cdot0361\pm\cdot0241$
F_5/F_4, $P_3/P_4(L)$	—	$S_6/S_7(L)$, O_4/O_1	—
$S_5/S_4(R)$, O_6/O_3	$-\cdot1053\pm\cdot0244$	$T_6/T_5(R)$, O_6/O_3	$-\cdot0252\pm\cdot0235$
$S_5/S_4(L)$, O_6/O_3	—	$T_6/T_5(L)$, O_6/O_3	—
F_3/F_4, $S_5/S_4(R)$	$-\cdot0807\pm\cdot0243$	$T_2/T_3(R)$, O_6/O_3	$-\cdot0247\pm\cdot0234$
F_3/F_4, $S_5/S_4(L)$	—	$T_2/T_3(L)$, O_6/O_3	—
$S_5/S_4(R)$, O_4/O_1	$-\cdot0786\pm\cdot0247$	F_3/F_4, $Ml_2/Ml_1(R)$	$-\cdot0239\pm\cdot0244$
$S_5/S_4(L)$, O_4/O_1	—	F_3/F_4, $Ml_2/Ml_1(L)$	—
F_5/F_4, $S_5/S_4(R)$	$-\cdot0764\pm\cdot0243$	$T_2/T_3(R)$, O_4/O_1	$+\cdot0231\pm\cdot0235$
F_5/F_4, $S_5/S_4(L)$	—	$T_2/T_3(L)$, O_4/O_1	—
F_5/F_4, $P_2/P_1(R)$	$-\cdot0733\pm\cdot0242$	$P_2/P_1(R)$, O_6/O_3	$+\cdot0227\pm\cdot0245$
F_5/F_4, $P_2/P_1(L)$	—	$P_2/P_1(L)$, O_6/O_3	—
F_3/F_4, $P_2/P_1(R)$	$-\cdot0726\pm\cdot0240$	$T_6/T_5(R)$, $O_5/(O_2+O_3)$	$-\cdot0220\pm\cdot0236$
F_3/F_4, $P_2/P_1(L)$	—	$T_6/T_5(L)$, $O_5/(O_2+O_3)$	—
$P_3/P_4(R)$, $O_5/(O_2+O_3)$	$+\cdot0707\pm\cdot0246$	$T_2/T_3(R)$, $O_5/(O_2+O_3)$	$-\cdot0121\pm\cdot0236$
$P_3/P_4(L)$, $O_5/(O_2+O_3)$	—	$T_2/T_3(L)$, $O_5/(O_2+O_3)$	—
F_3/F_4, $P_3/P_4(R)$	$+\cdot0704\pm\cdot0241$	F_5/F_4, $S_6/S_7(R)$	$+\cdot0113\pm\cdot0237$
F_3/F_4, $P_3/P_4(L)$	—	F_5/F_4, $S_6/S_7(L)$	—
$P_2/P_1(R)$, O_4/O_1	$-\cdot0663\pm\cdot0244$	$P_2/P_1(R)$, $O_5/(O_2+O_3)$	$-\cdot0089\pm\cdot0246$
$P_2/P_1(L)$, O_4/O_1	—	$P_2/P_1(L)$, $O_5/(O_2+O_3)$	—
$P_3/P_4(R)$, O_6/O_3	$+\cdot0662\pm\cdot0244$	$S_6/S_7(R)$, $O_5/(O_2+O_3)$	$-\cdot0087\pm\cdot0239$
$P_3/P_4(L)$, O_6/O_3	—	$S_6/S_7(L)$, $O_5/(O_2+O_3)$	—
$T_6/T_5(R)$, O_4/O_1	$+\cdot0609\pm\cdot0235$	$S_6/S_7(R)$, O_6/O_3	$-\cdot0066\pm\cdot0238$
$T_6/T_5(L)$, O_4/O_1	—	$S_6/S_7(L)$, O_6/O_3	—

As it does not seem possible to lay any stress on contiguity as differentiating the correlations given in Tables XVI and XVII, we will pool the 40 correlations, which then provide a mean numerical correlation of ·0558, and simply ask whether these 40 correlations of lateral with non-lateral indices may be taken as a random sample from parent populations with a correlation coefficient of $r=\cdot0558$. This inquiry, if very laborious, will enable us at any rate to say whether it is reasonable to

represent by a single value of r the influence of non-lateral indices on lateral indices. The results are as follows:

Probability Integrals: From left to right

$$\log \lambda_n = -15{\cdot}894{,}7938,$$

$$\sqrt{n}\log_{10} e = 2{\cdot}746{,}71958,$$

$$u = -\frac{\log_{10}\lambda_n}{\sqrt{n}\log_{10} e} = 5{\cdot}786{,}828,$$

From right to left

$$\log \lambda'_n = -16{\cdot}017{,}2577,$$

$$\sqrt{n}\log_{10} e = 2{\cdot}746{,}71958,$$

$$u' = -\frac{\log_{10}\lambda'_n}{\sqrt{n}\log_{10} e} = 5{\cdot}831{,}4135,$$

$$P_{\lambda_n} = I(n-1, u) = I(39, 5{\cdot}786{,}828), \quad P_{\lambda'_n} = I(n-1, u') = I(39, 5{\cdot}831{,}4135).$$

Hence from the *Incomplete* Γ-*Function Tables*:

$$P_{\lambda_n} = {\cdot}3083, \qquad P_{\lambda'_n} = {\cdot}3251,$$

$$Q_{\lambda_n} = {\cdot}6917, \qquad Q_{\lambda'_n} = {\cdot}6749.$$

The latter is somewhat the more stringent. It indicates that in 100 random samples from the population of correlation coefficients with mean ·0558, 67 would give a more improbable result than our system of correlations of indices of lateral and non-lateral bones. We do not assert that our system *is* such a selection, but we do assert that it is a *highly* reasonable description of what has been observed, that is to say a successful graduation. In other words the relation of lateral and non-lateral indices can be expressed by the single number ·0558.

(13) *Correlation of two Indices of the same Bone.*

This is our Group (vi) and the correlations falling under this heading are given in Table XVIII below:

TABLE XVIII. *Correlation of two Indices of the same Bone.*

Index Pair			Correlation and p.e.	Index Pair			Correlation and p.e.
(a_1)	O_4/O_1,	$O_8/O_7\,(R)$	$+{\cdot}6672 \pm {\cdot}0129$	(i_1)	$S_6/S_7\,(R)$,	$S_5/S_4\,(R)$	$+{\cdot}1633 \pm {\cdot}0247$
(a_2)	O_4/O_1,	$O_8/O_7\,(L)$	$+{\cdot}6042 \pm {\cdot}0147$	(j_1)	$P_3/P_4\,(R)$,	$P_2/P_1\,(R)$	$-{\cdot}1133 \pm {\cdot}0239$
(b)	F_5/F_4,	F_3/F_4	$+{\cdot}5066 \pm {\cdot}0168$	(k)	$O_5/(O_2+O_3)$,	O_4/O_1	$-{\cdot}1109 \pm {\cdot}0231$
(c)	$O_5/(O_2+O_3)$,	O_6/O_3	$+{\cdot}4764 \pm {\cdot}0180$	(l)	O_4/O_1,	O_6/O_3	$-{\cdot}0894 \pm {\cdot}0233$
(d_1)	O_4/O_1,	$O_8/O_9\,(R)$	$+{\cdot}3744 \pm {\cdot}0201$	(m_1)	$O_8/O_9\,(R)$,	O_6/O_3	$-{\cdot}0865 \pm {\cdot}0232$
(d_2)	O_4/O_1,	$O_8/O_9\,(L)$	$+{\cdot}3130 \pm {\cdot}0211$	(j_2)	$P_3/P_4\,(L)$,	$P_2/P_1\,(L)$	$-{\cdot}0753 \pm {\cdot}0241$
(e_1)	$O_8/O_9\,(R)$,	$O_8/O_7\,(R)$	$+{\cdot}3094 \pm {\cdot}0210$	(n_1)	$T_6/T_5\,(R)$,	$T_2/T_3\,(R)$	$+{\cdot}0679 \pm {\cdot}0233$
(f_1)	$F_2/F_1\,(R)$,	F_3/F_4	$+{\cdot}2677 \pm {\cdot}0210$	(o_1)	$O_5/(O_2+O_3)$,	$O_8/O_9\,(R)$	$-{\cdot}0667 \pm {\cdot}0233$
(e_2)	$O_8/O_9\,(L)$,	$O_8/O_7\,(L)$	$+{\cdot}2573 \pm {\cdot}0216$	(m_2)	$O_8/O_9\,(L)$,	O_6/O_3	$-{\cdot}0584 \pm {\cdot}0233$
(f_2)	$F_2/F_1\,(L)$,	F_3/F_4	$+{\cdot}2566 \pm {\cdot}0212$	(p_2)	O_6/O_3,	$O_8/O_7\,(L)$	$-{\cdot}0549 \pm {\cdot}0232$
(g_2)	$Mx_2/Mx_1\,(L)$,	$Mx_3/Mx_1\,(L)$	$+{\cdot}2284 \pm {\cdot}0303$	(n_2)	$T_6/T_5\,(L)$,	$T_2/T_3\,(L)$	$+{\cdot}0460 \pm {\cdot}0233$
(g_1)	$Mx_2/Mx_1\,(R)$,	$Mx_3/Mx_1\,(R)$	$+{\cdot}2103 \pm {\cdot}0311$	(q_2)	$O_5/(O_2+O_3)$,	$O_8/O_7\,(L)$	$+{\cdot}0416 \pm {\cdot}0233$
(h_2)	F_5/F_4,	$F_2/F_1\,(L)$	$+{\cdot}2062 \pm {\cdot}0218$	(q_1)	$O_5/(O_2+O_3)$,	$O_8/O_7\,(R)$	$+{\cdot}0327 \pm {\cdot}0233$
(i_2)	$S_6/S_7\,(L)$,	$S_5/S_4\,(L)$	$+{\cdot}1909 \pm {\cdot}0242$	(p_1)	O_6/O_3,	$O_8/O_7\,(R)$	$-{\cdot}0135 \pm {\cdot}0234$
(h_1)	F_5/F_4,	$F_2/F_1\,(R)$	$+{\cdot}1786 \pm {\cdot}0220$	(o_2)	$O_5/(O_2+O_3)$,	$O_8/O_9\,(L)$	$-{\cdot}0032 \pm {\cdot}0234$

The small letters in the corners of Columns 1 and 3 indicate pairs where they exist, the subscript 1 for right and 2 for left.

It will be seen at once that indices on the same bone present a wide range of correlation values. But the source of these differences is fairly easy of analysis. Thus:

(*a*) $O_4 = O_8(R) + O_8(L)$ and there will be a high positive correlation between $O_8(R)$ or $O_8(L)$ and O_4 if there be little between the other occipital measurements involved.

(*b*) F_4 is the denominator of both indices, hence if there be little correlation between the other frontal lengths (*b*) should show a considerable positive correlation.

(*c*) O_3 is common to both index denominators and this should involve positive correlation in the indices.

(*d*) The same condition holds as in (*a*) and involves positive correlation.

(*e*) O_8 is the numerator of both indices and positive correlation follows.

(*f*) and (*h*) F_4 is nearly parallel to F_1, and so to speak covers it, so that F_1 can be almost looked upon as a part of F_4, and positive index correlation may be anticipated.

(*g*) Mx_1 is the common denominator of both fractions, and we may anticipate positive correlation.

(*i*) S_4 overlies S_7, or the latter may be looked upon as part of the former; positive correlation will result.

(*j*) P_3 and P_1 probably possess considerable correlation, so that a negative correlation would arise between the (*j*) indices.

(*k*) and (*l*) O_1 is likely to have a negative correlation with both O_2 and O_3, thus productive of a negative correlation in the (*k*) and (*l*) indices.

(*m*) to (*q*) In the remaining indices one does not see any special reason for their magnitude or sign.

From the slender material we have, we may reason that this system of correlations of indices of the same bone is more or less capable of interpretation, if a high correlation arises when the two indices have a common factor, a somewhat lower correlation when a factor of one overlaps a factor of the other, and little or no correlation when neither of these cases can be seen to hold. Until the correlations of absolute lengths on the same bone have been worked out, it is not possible to ascertain the limits of the truth of this statement. But we may consider whether the 10 correlations in the *m*—*q* group could be reasonably considered as a random selection from a parent population of zero correlation. The values of the correlations in these groups are:

$-\cdot0865 \pm \cdot0232$,	$+\cdot0460 \pm \cdot0233$,
$+\cdot0679 \pm \cdot0233$,	$+\cdot0416 \pm \cdot0233$,
$-\cdot0667 \pm \cdot0233$,	$+\cdot0327 \pm \cdot0233$,
$-\cdot0584 \pm \cdot0233$,	$-\cdot0135 \pm \cdot0234$,
$-\cdot0549 \pm \cdot0232$,	$-\cdot0032 \pm \cdot0234$.

Mean of the six negative correlations $= -\cdot0472$,
Mean of the four positive correlations $= \cdot0471$.

The probable error can be taken as ·0233 for the system, or the standard error as ·034545. Working out the probability integrals and their logarithms, we have:

$$\log_{10}\lambda_n = -7{\cdot}358{,}4735, \qquad \log_{10}\lambda'_n = -4{\cdot}868{,}7097,$$

$$\sqrt{n}\log_{10}e = \sqrt{10}\times{\cdot}434{,}2945 = 1{\cdot}373{,}3598,$$

$$u = 5{\cdot}358{,}0085, \qquad u' = 3{\cdot}545{,}1086,$$

$$P_{\lambda_n} = I(9,\ 5{\cdot}358{,}0085) = {\cdot}9731, \qquad P_{\lambda'_n} = I(9,\ 3{\cdot}545{,}1086) = {\cdot}6818,$$

$$Q_{\lambda_n} = {\cdot}0269, \qquad Q_{\lambda'_n} = {\cdot}3182.$$

Notwithstanding the value of $Q_{\lambda'_n}$, which would warrant the "reasonableness" of the hypothesis, we feel bound to take the more stringent Q_{λ_n} for our guide, or more probable distributions of the correlation values would occur in 97 % of random samples than in the sample actually observed if they were taken from a population of zero correlation.

Now there are those who accept a < ·01 probability as justifying the rejection of an hypothesis; others demand a < ·05 probability. We hold that in each case the actual probability should be determined, and then judgment formed in relation to the material on which it is based. In our case, we do not think that $Q_{\lambda_n} = {\cdot}03$ about, justifies us in accepting the hypothesis that, if we exclude cases of common factors in the indices and "overlapping," there is little or no correlation of indices on the same bone. We prefer to wait until the correlations of absolute lengths on the same bone have been measured, and we can trace the contribution of each absolute length to the make-up of the index correlations.

What we feel confidence in asserting is that common factors, and, to a less extent, "overlaps," do increase the numerical magnitude of the correlation of indices of the same bone.

(14) *Influence of Homology on Indices taken on like Bones.*

This section will deal with our Group (viii). Unfortunately, the number of correlations falling under this heading and already worked out are few. They are given in Table XIX below.

In this table, beside the correlations of heterologous indices on homologous bones, we have the correlations of heterologous indices on the same bone, and further the homologous indices of homologous bones. It would be of great value were we able to deduce approximate values for the correlations in Column 2 of Table XIX from the data given in Columns 4, 6 and 8, that is to say if we could deduce the cross-correlations from our knowledge of direct correlations and the correlations for homology.

Formulae, having at least some theoretical basis, have been suggested with this end in view. Let A and B be two characters, and the subscripts R and L refer to the right and left sides of the skull, then we will test the following relations:

$$\left.\begin{aligned} r_{A_RB_L} &= \tfrac{1}{2}\left(r_{A_RB_R}\times r_{B_RB_L} + r_{B_LA_L}\times r_{A_LA_R}\right) \\ r_{A_LB_R} &= \tfrac{1}{2}\left(r_{A_LB_L}\times r_{B_LB_R} + r_{A_RB_R}\times r_{A_RA_L}\right) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(\epsilon).$$

TABLE XIX.

Correlation of Heterologous Indices on a pair of Homologous Bones.

Heterologous Indices on Homologous Bones		Heterologous Indices on same Bone		Homologous Bones, same Index		Homologous Bones, same Index	
Index Pair	Correlation and p.e.	Index Pair	Correlation and p.e.	Index Pair	Correlation and p.e.	Index Pair	Correlation and p.e.
$S_6/S_7\,(R)$, $S_5/S_4\,(L)$	$+\cdot1787\pm\cdot0244$	$S_6/S_7\,(R)$, $S_5/S_4\,(R)$	$+\cdot1633\pm\cdot0247$	$S_6/S_7\,(R)$, $S_6/S_7\,(L)$	$+\cdot9048\pm\cdot0043$	$S_5/S_4\,(R)$, $S_5/S_4\,(L)$	$+\cdot8822\pm\cdot0056$
$S_6/S_7\,(L)$, $S_5/S_4\,(R)$	$+\cdot1863\pm\cdot0245$	$S_6/S_7\,(L)$, $S_5/S_4\,(L)$	$+\cdot1909\pm\cdot0242$	—	—	—	—
$T_6/T_5\,(R)$, $T_2/T_3\,(L)$	$+\cdot0322\pm\cdot0234$	$T_6/T_5\,(R)$, $T_2/T_3\,(R)$	$+\cdot0679\pm\cdot0233$	$T_6/T_5\,(R)$, $T_6/T_5\,(L)$	$+\cdot6185\pm\cdot0145$	$T_2/T_3\,(R)$, $T_2/T_3\,(L)$	$+\cdot7934\pm\cdot0086$
$T_6/T_5\,(L)$, $T_2/T_3\,(R)$	$+\cdot0415\pm\cdot0234$	$T_6/T_5\,(L)$, $T_2/T_3\,(L)$	$+\cdot0460\pm\cdot0233$	—	—	—	—
$P_3/P_4\,(R)$, $P_2/P_1\,(L)$	$-\cdot1169\pm\cdot0245$	$P_3/P_4\,(R)$, $P_2/P_1\,(R)$	$-\cdot1133\pm\cdot0239$	$P_3/P_4\,(R)$, $P_3/P_4\,(L)$	$+\cdot4057\pm\cdot0207$	$P_2/P_1\,(R)$, $P_2/P_1\,(L)$	$+\cdot7892\pm\cdot0093$
$P_3/P_4\,(L)$, $P_2/P_1\,(R)$	$-\cdot1756\pm\cdot0240$	$P_3/P_4\,(L)$, $P_2/P_1\,(L)$	$-\cdot0753\pm\cdot0241$	—	—	—	—
$O_8/O_9\,(R)$, $O_8/O_7\,(L)$	$+\cdot0700\pm\cdot0231$	$O_8/O_9\,(R)$, $O_8/O_7\,(R)$	$+\cdot3094\pm\cdot0210$	$O_8/O_9\,(R)$, $O_8/O_9\,(L)$	$+\cdot4769\pm\cdot0179$	$O_8/O_7\,(R)$, $O_8/O_7\,(L)$	$+\cdot6044\pm\cdot0146$
$O_8/O_9\,(L)$, $O_8/O_7\,(R)$	$+\cdot0584\pm\cdot0231$	$O_8/O_9\,(L)$, $O_8/O_7\,(L)$	$+\cdot2573\pm\cdot0216$	—	—	—	—
$Mx_2/Mx_1\,(R)$, $Mx_3/Mx_1\,(L)$	$+\cdot1857\pm\cdot0316$	$Mx_2/Mx_1\,(R)$, $Mx_3/Mx_1\,(R)$	$+\cdot2103\pm\cdot0311$	$Mx_2/Mx_1\,(R)$, $Mx_3/Mx_1\,(L)$	$+\cdot9344\pm\cdot0038$	$Mx_3/Mx_1\,(R)$, $Mx_3/Mx_1\,(L)$	$+\cdot7894\pm\cdot0125$
$Mx_2/Mx_1\,(L)$, $Mx_3/Mx_1\,(R)$	$+\cdot1757\pm\cdot0324$	$Mx_2/Mx_1\,(L)$, $Mx_3/Mx_1\,(L)$	$+\cdot2284\pm\cdot0303$	—	—	—	—

The observed values are set against the values deduced from the above formulae in the following table:

TABLE XX. *Observed Cross-Correlations compared with Values computed from* (ϵ).

A	B	Observed r_{AB}	Computed r_{AB}	Remarks
$S_6/S_7\,(R)$,	$S_5/S_4\,(L)$	$+\cdot1787\pm\cdot0244$	$+\cdot1584$	Observed in excess, but compatible considering the probable errors
$S_6/S_7\,(L)$,	$S_5/S_4\,(R)$	$+\cdot1863\pm\cdot0245$	$+\cdot1581$	
$T_6/T_5\,(R)$,	$T_2/T_3\,(L)$	$+\cdot0322\pm\cdot0234$	$+\cdot0412$	Observed and computed, quite compatible
$T_6/T_5\,(L)$,	$T_2/T_3\,(R)$	$+\cdot0415\pm\cdot0234$	$+\cdot0392$	
$P_3/P_4\,(R)$,	$P_2/P_1\,(L)$	$-\cdot1169\pm\cdot0245$	$-\cdot0600$	Observed in excess of computed, and differences pointing towards significance
$P_3/P_4\,(L)$,	$P_2/P_1\,(R)$	$-\cdot1756\pm\cdot0240$	$-\cdot0525$	
$O_8/O_9\,(R)$,	$O_8/O_7\,(L)$	$+\cdot0700\pm\cdot0231$	$+\cdot1548^5$	Observed in defect of computed, and differences pointing towards significance
$O_8/O_9\,(L)$,	$O_8/O_7\,(R)$	$+\cdot0584\pm\cdot0231$	$+\cdot1515$	
$Mx_2/Mx_1\,(R)$,	$Mx_3/Mx_1\,(L)$	$+\cdot1857\pm\cdot0316$	$+\cdot1892$	Observed in defect, but differences so slight as to render the results compatible
$Mx_2/Mx_1\,(L)$,	$Mx_3/Mx_1\,(R)$	$+\cdot1757\pm\cdot0324$	$+\cdot1884$	

The differences between the observed and computed correlations run first one way and then the other. The worst cases are those for the parietal and the occipital indices, but while for the former the observed are considerably in excess, in the latter the observed are considerably in defect. The results are on the whole not encouraging for the value of the formulae (ϵ).

The formulae cannot, however, be at once dismissed as not even approximate. It seemed worth while testing the formulae (ϵ) on a wider field. Table XXI is accordingly given here, although it belongs to another section of our work* on morphometric characters, namely, to that dealing with the correlations of heterologous absolute lengths.

Table XXI gives 120 correlations of absolute lengths, enough material to test formulae (ϵ) on 56 cases of absolute measurements. We do not intend in the present paper to analyse this material, but we may draw the reader's attention to one or two points having a bearing on our present inquiry.

Putting aside the Homologous Correlations, there are only four out of 28 double cases, where either direct or cross-correlation exceeds ·35. These are F_1, F_2 (about ·64), F_1, P_2 (about ·54), F_2, P_2 (about ·46) and T_2, S_5 (about ·37). On examination of our diagrams (pp. 425, 426) the reader will see some reason for these relatively high correlations. But he will probably be surprised to see how low the correlations on the whole are; out of 112 non-homologous correlations, 96 are below ·25 in value! It follows that the factor of size is not nearly as important as that of homology in the relationship of different parts of the skull. The last two columns of Table XXI

* From the 63 original measurements, nearly 2000 correlation tables could be formed—ample material for a further section of this paper on correlation of absolute lengths of individual bones of the skull. Only some 120 are given in Table XXI.

TABLE XXI. *Correlation Coefficients of Absolute Measurements.*

Characters		Uni-lateral Correlations	Uni-lateral Correlations	Cross-Correlations	Cross-Correlations	Homologous Correlations	Homologous Correlations
A	*B*	$r_{A_R B_R}$	$r_{A_L B_L}$	$r_{A_R B_L}$	$r_{A_L B_R}$	$r_{A_R A_L}$	$r_{B_R B_L}$
F_1	F_2	·64459 ± ·01324 (887)	·64490 ± ·01325 (884)	·63135 ± ·01363 (885)	·62401 ± ·01383 (887)	·9059 ± ·0041 (885)	·9384 ± ·0027 (887)
F_1	P_2	·54411 ± ·01682 (797)	·55331 ± ·01664 (791)	·50949 ± ·01777 (790)	·48849 ± ·01821 (796)	·9059 ± ·0041 (885)	·6738 ± ·0134 (754)
F_1	O_7	·16786 ± ·02227 (866)	·14901 ± ·02238 (868)	·17235 ± ·02221 (868)	·15460 ± ·02237 (866)	·9059 ± ·0041 (885)	·7954 ± ·0084 (864)
F_1	T_2	·20519 ± ·02199 (863)	·17773 ± ·02215 (870)	·18081 ± ·02213 (869)	·18295 ± ·02219 (863)	·9059 ± ·0041 (885)	·8145 ± ·0078 (856)
F_1	S_5	·21258 ± ·02325 (767)	·18791 ± ·02318 (788)	·19203 ± ·02313 (789)	·19635 ± ·02343 (766)	·9059 ± ·0041 (885)	·7962 ± ·0092 (722)
F_1	Ml_1	·14707 ± ·02377 (771)	·15323 ± ·02364 (776)	·16192 ± ·02358 (776)	·14376 ± ·02379 (771)	·9059 ± ·0041 (885)	·9399 ± ·0029 (718)
F_1	Mx_1	·18932 ± ·02407 (730)	·19628 ± ·02410 (724)	·20226 ± ·02405 (723)	·20191 ± ·02393 (731)	·9059 ± ·0041 (885)	·9766 ± ·0012 (692)
F_2	P_2	·44013 ± ·01860 (855)	·46383 ± ·01886 (788)	·46949 ± ·01872 (789)	·43063 ± ·01950 (794)	·9384 ± ·0027 (887)	·6738 ± ·0134 (754)
F_2	O_7	·09221 ± ·02277 (863)	·09242 ± ·02275 (864)	·11462 ± ·02259 (868)	·07784 ± ·02282 (863)	·9384 ± ·0027 (887)	·7954 ± ·0084 (864)
F_2	T_2	·14002 ± ·02252 (864)	·12279 ± ·02257 (866)	·11099 ± ·02263 (867)	·14882 ± ·02248 (861)	·9384 ± ·0027 (887)	·8145 ± ·0078 (856)
F_2	S_5	·05944 ± ·02430 (765)	·04764 ± ·02401 (786)	·04933 ± ·02397 (788)	·06783 ± ·02431 (763)	·9384 ± ·0027 (887)	·7962 ± ·0092 (722)
F_2	Ml_1	·04078 ± ·02430 (768)	·08813 ± ·02405 (774)	·08516 ± ·02405 (775)	·06631 ± ·02423 (768)	·9384 ± ·0027 (887)	·9399 ± ·0029 (718)
F_2	Mx_1	·09572 ± ·02473 (730)	·08287 ± ·02491 (723)	·12021 ± ·02472 (723)	·06292 ± ·02485 (728)	·9384 ± ·0027 (887)	·9766 ± ·0012 (692)
P_2	O_7	·21141 ± ·02310 (778)	·22452 ± ·02307 (771)	·22557 ± ·02295 (778)	·19277 ± ·02339 (771)	·6738 ± ·0134 (754)	·7954 ± ·0084 (864)
P_2	T_2	·01113 ± ·02410 (783)	− ·00913 ± ·02412 (782)	− ·00915 ± ·02407 (785)	·03169 ± ·02426 (772)	·6738 ± ·0134 (754)	·8145 ± ·0078 (856)
P_2	S_5	− ·05912 ± ·02470 (740)	− ·04921 ± ·02454 (752)	− ·01640 ± ·02484 (737)	− ·06682 ± ·02519 (711)	·6738 ± ·0134 (754)	·7962 ± ·0092 (722)
P_2	Ml_1	·07406 ± ·02539 (698)	·07297 ± ·02531 (703)	·06225 ± ·02530 (705)	·08769 ± ·02535 (697)	·6738 ± ·0134 (754)	·9399 ± ·0029 (718)
P_2	Mx_1	·10606 ± ·02601 (657)	·11425 ± ·02612 (650)	·12011 ± ·02608 (650)	·09814 ± ·02608 (656)	·6738 ± ·0134 (754)	·9766 ± ·0012 (692)
O_7	T_2	·08399 ± ·02306 (844)	·05754 ± ·02304 (851)	·06056 ± ·02308 (848)	·08433 ± ·02303 (846)	·7954 ± ·0084 (864)	·8145 ± ·0078 (856)
O_7	S_5	·02986 ± ·02464 (748)	·02113 ± ·02427 (772)	·04268 ± ·02430 (768)	·02957 ± ·02463 (749)	·7954 ± ·0084 (864)	·7962 ± ·0092 (722)
O_7	Ml_1	·05668 ± ·02450 (753)	·04960 ± ·02441 (760)	·05372 ± ·02443 (758)	·05470 ± ·02449 (754)	·7954 ± ·0084 (864)	·9399 ± ·0029 (718)
O_7	Mx_1	·14721 ± ·02464 (717)	·14950 ± ·02478 (708)	·14497 ± ·02484 (707)	·17333 ± ·02443 (717)	·7954 ± ·0084 (864)	·9766 ± ·0012 (692)
T_2	S_5	·37646 ± ·02105 (756)	·43502 ± ·01957 (781)	·35829 ± ·02110 (776)	·37283 ± ·02112 (756)	·8145 ± ·0078 (856)	·7962 ± ·0092 (722)
T_2	Ml_1	·08289 ± ·02431 (759)	·14093 ± ·02389 (766)	·09582 ± ·02425 (760)	·11448 ± ·02413 (761)	·8145 ± ·0078 (856)	·9399 ± ·0029 (718)
T_2	Mx_1	·13092 ± ·02472 (719)	·12254 ± ·02486 (714)	·09550 ± ·02505 (712)	·12626 ± ·02474 (720)	·8145 ± ·0078 (856)	·9766 ± ·0012 (692)
S_5	Ml_1	·16014 ± ·02545 (667)	·20571 ± ·02447 (697)	·15571 ± ·02527 (678)	·17720 ± ·02480 (694)	·7962 ± ·0092 (722)	·9399 ± ·0029 (718)
S_5	Mx_1	·20490 ± ·02568 (633)	·24992 ± ·02490 (645)	·21501 ± ·02570 (627)	·21501 ± ·02570 (627)	·7962 ± ·0092 (722)	·9766 ± ·0012 (692)
Ml_1	Mx_1	·18177 ± ·02541 (661)	·16939 ± ·02556 (657)	·17765 ± ·02554 (654)	·17765 ± ·02554 (654)	·9399 ± ·0029 (718)	·9766 ± ·0012 (692)
Numerical Mean		·17484	·17969	·17262	·17317	—	—

show us that the lowest homologous correlation is ·6738, and they fill the range from that value up to ·9766. But the highest absolute size non-homologous correlation is ·6449, and they range from this numerically down to ·0091. The only certain *negative* correlation is that between the parietal arc P_2 from bregma to sphenion, and the sphenoidal chord from krotaphion to the postreme point of the sphenoid as exposed on the base of the skull.

Thus it would seem that the factor of size as judged by the correlation of absolute lengths of the cranial bones, while greater than the factor of contiguity, is still very much less intense than the factor of homology.

While it may be shown from Columns 3 and 4 of Table XXI that correlations of absolute lengths on the left side, $r_{A_LB_L}$, do not as a whole differ significantly from those of absolute length on the right side, $r_{A_RB_R}$, yet the cross-correlations $r_{A_RB_L}$ and $r_{A_LB_R}$ are smaller, but while giving considerable differences (see Table XXII, Columns 3 and 4), they are yet again within the limits of random sampling. As the cross-correlations are less than the direct correlations, we are justified in seeing what values formulae (ϵ) will provide for the cross-correlations. The computed values are given in Column 5 of Table XXII. They show at once that the differences of $r_{A_RB_L}$ and $r_{A_LB_R}$ in Column 7 are wholly trivial and thus unlike the differences in Column 4. Thus, whether the calculated cross-correlations do or do not equal the observed values, the numerical values of

$$\tfrac{1}{2}(r_{A_RB_R} \times r_{B_RB_L} + r_{B_LA_L} \times r_{A_LA_R})$$

and

$$\tfrac{1}{2}(r_{A_LB_I} \times r_{B_LB_R} + r_{B_RA_R} \times r_{A_RA_L})$$

are for practical purposes equal.

On the other hand, if we compare the means of the observed cross-correlations $r_{A_RB_L}$ and $r_{A_LB_R}$ with the means of the corresponding values computed by (ϵ), the latter are in all the 28 cases smaller with the one exception of T_2 and Mx_1, where (ϵ) gives a value larger by ·0026.

It is clear, therefore, that whatever the index correlations may do, (ϵ) does not represent a general principle applying also to the correlations of absolute lengths. If O_m be an observed cross-correlation mean, i.e. $O_m = \frac{1}{2}(O_{R,L} + O_{L,R})$, and C_m be a computed cross-correlation mean, i.e. $C_m = \frac{1}{2}(C_{R,L} + C_{L,R})$, we have calculated the linear relation of O_m to C_m as given by Least Squares and we find the equation

$$O_m = 1{\cdot}098{,}138\, C_m + {\cdot}005{,}998 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\epsilon'),$$

where the constant term may be taken as ·0060.

The results are exhibited in Table XXIII.

It will be seen that the mean numerical difference between the observed and computed means is (·0094), and this is less than half the difference between the observed $r_{A_RB_L}$ and $r_{A_LB_R}$. Hence, if we suppose the difference between the R, L and L, R values of the cross-correlations to be merely due to random sampling, not to inherent right- and left-sidedness, it follows that formulae (ϵ') will provide, well

TABLE XXII.

Difference of Observed and Computed Cross-Correlations of Absolute Lengths.

Index Pair	Observed Correlations	Mean R, L and L, R	Difference $R, L - L, R$	Computed Correlations (ϵ)	Mean R, L and L, R	Difference $R, L - L, R$
$F_1(R)$, $F_2(L)$	·6314 ± ·0136	·6277	+ ·0074	·5946	·5945	+ ·0001
$F_1(L)$, $F_2(R)$	·6240 ± ·0138			·5945		
$F_1(R)$, $P_2(L)$	·5095 ± ·0178	·4990	+ ·0210	·4339	·4334	+ ·0010
$F_1(L)$, $P_2(R)$	·4885 ± ·0182			·4329		
$F_1(R)$, $O_7(L)$	·1724 ± ·0222	·1635	+ ·0178	·1343	·1348	− ·0010
$F_1(L)$, $O_7(R)$	·1546 ± ·0224			·1353		
$F_1(R)$, $T_2(L)$	·1808 ± ·0221	·1819	− ·0022	·1641	·1647	− ·0012
$F_1(L)$, $T_2(R)$	·1830 ± ·0222			·1653		
$F_1(R)$, $S_5(L)$	·1920 ± ·0231	·1942	− ·0044	·1697	·1704	− ·0014
$F_1(L)$, $S_5(R)$	·1964 ± ·0234			·1711		
$F_1(R)$, $Ml_1(L)$	·1619 ± ·0236	·1528	+ ·0181	·1385	·1385	− ·0001
$F_1(L)$, $Ml_1(R)$	·1438 ± ·0238			·1386		
$F_1(R)$, $Mx_1(L)$	·2023 ± ·0241	·2021	+ ·0004	·1813	·1815	− ·0003
$F_1(L)$, $Mx_1(R)$	·2019 ± ·0239			·1816		
$F_2(R)$, $P_2(L)$	·4695 ± ·0187	·4501	+ ·0389	·3659	·3643	+ ·0031
$F_2(L)$, $P_2(R)$	·4306 ± ·0195			·3628		
$F_2(R)$, $O_7(L)$	·1146 ± ·0226	·0962	+ ·0368	·0800	·0800	·0000
$F_2(L)$, $O_7(R)$	·0778 ± ·0228			·0800		
$F_2(R)$, $T_2(L)$	·1110 ± ·0226	·1299	− ·0378	·1146	·1152	− ·0011
$F_2(L)$, $T_2(R)$	·1488 ± ·0225			·1157		
$F_2(R)$, $S_5(L)$	·0493 ± ·0240	·0586	− ·0185	·0460	·0465	− ·0009
$F_2(L)$, $S_5(R)$	·0678 ± ·0243			·0469		
$F_2(R)$, $Ml_1(L)$	·0852 ± ·0241	·0757	+ ·0189	·0605	·0605	− ·0001
$F_2(L)$, $Ml_1(R)$	·0663 ± ·0242			·0606		
$F_2(R)$, $Mx_1(L)$	·1202 ± ·0247	·0916	+ ·0573	·0856	·0855	+ ·0002
$F_2(L)$, $Mx_1(R)$	·0629 ± ·0249			·0854		
$P_2(R)$, $O_7(L)$	·2256 ± ·0230	·2092	+ ·0328	·1597	·1601	− ·0008
$P_2(L)$, $O_7(R)$	·1928 ± ·0234			·1605		
$P_2(R)$, $T_2(L)$	− ·0092 ± ·0241	·0113	− ·0409	·0015	·0007	+ ·0015
$P_2(L)$, $T_2(R)$	·0317 ± ·0243			·0000		
$P_2(R)$, $S_5(L)$	− ·0164 ± ·0248	− ·0416	− ·0504	− ·0401	− ·0398	+ ·0006
$P_2(L)$, $S_5(R)$	− ·0668 ± ·0252			− ·0395		
$P_2(R)$, $Ml_1(L)$	·0623 ± ·0253	·0750	− ·0254	·0594	·0593	+ ·0002
$P_2(L)$, $Ml_1(R)$	·0877 ± ·0254			·0592		
$P_2(R)$, $Mx_1(L)$	·1201 ± ·0261	·1091	+ ·0220	·0903	·0909	− ·0012
$P_2(L)$, $Mx_1(R)$	·0981 ± ·0261			·0915		
$O_7(R)$, $T_2(L)$	·0606 ± ·0231	·0724	− ·0237	·0571	·0574	− ·0005
$O_7(L)$, $T_2(R)$	·0843 ± ·0230			·0576		
$O_7(R)$, $S_5(L)$	·0427 ± ·0243	·0361	+ ·0131	·0203	·0203	·0000
$O_7(L)$, $S_5(R)$	·0296 ± ·0246			·0203		
$O_7(R)$, $Ml_1(L)$	·0537 ± ·0244	·0542	− ·0010	·0464	·0461	+ ·0005
$O_7(L)$, $Ml_1(R)$	·0547 ± ·0245			·0459		
$O_7(R)$, $Mx_1(L)$	·1450 ± ·0248	·1592	− ·0283	·1313	·1314	− ·0002
$O_7(L)$, $Mx_1(R)$	·1733 ± ·0244			·1315		
$T_2(R)$, $S_5(L)$	·3583 ± ·0211	·3656	− ·0145	·3270	·3267	+ ·0005
$T_2(L)$, $S_5(R)$	·3728 ± ·0211			·3265		
$T_2(R)$, $Ml_1(L)$	·0958 ± ·0243	·1052	− ·0187	·0963	·0982	− ·0037
$T_2(L)$, $Ml_1(R)$	·1145 ± ·0241			·1000		
$T_2(R)$, $Mx_1(L)$	·0955 ± ·0251	·1109	− ·0308	·1138	·1135	+ ·0006
$T_2(L)$, $Mx_1(R)$	·1263 ± ·0247			·1132		
$S_5(R)$, $Ml_1(L)$	·1557 ± ·0253	·1665	− ·0215	·1572	·1588	− ·0032
$S_5(L)$, $Ml_1(R)$	·1772 ± ·0248			·1604		
$S_5(R)$, $Mx_1(L)$	·2150 ± ·0257	·2150	·0000	·1995	·2015	− ·0041
$S_5(L)$, $Mx_1(R)$	·2150 ± ·0257			·2036		
$Ml_1(R)$, $Mx_1(L)$	·1776 ± ·0255	·1776	− ·0001	·1684	·1683	+ ·0002
$Ml_1(L)$, $Mx_1(R)$	·1777 ± ·0255			·1681		

TABLE XXIII.

Comparison of Observed and Computed (by (ϵ')) Cross-Correlations.

Index Pair	Observed Mean O_m of R, L and L, R	Difference between observed R, L and L, R	Difference between observed Mean and Mean computed by (ϵ')	Computed Mean C_m R, L and L, R by (ϵ')
F_1, F_2	·6277	+·0074	−·0311	·6588
F_1, P_2	·4990	+·0210	+·0121	·4869
F_1, O_7	·1635	+·0178	+·0095	·1540
F_1, T_2	·1819	−·0022	−·0049	·1868
F_1, S_5	·1942	−·0044	+·0011	·1931
F_1, Ml_1	·1528	+·0181	−·0053	·1581
F_1, Mx_1	·2021	+·0004	−·0032	·2053
F_2, P_2	·4501	+·0389	+·0440	·4061
F_2, O_7	·0962	+·0368	+·0023	·0939
F_2, T_2	·1299	−·0378	−·0026	·1325
F_2, S_5	·0586	−·0185	+·0015	·0571
F_2, Ml_1	·0757	+·0189	+·0033	·0724
F_2, Mx_1	·0916	+·0573	−·0083	·0999
P_2, O_7	·2092	+·0328	+·0274	·1818
P_2, T_2	·0113	−·0409	+·0045	·0068
P_2, S_5	−·0416	−·0504	+·0039	−·0377
P_2, Ml_1	·0750	−·0254	+·0039	·0711
P_2, Mx_1	·1091	+·0220	+·0033	·1058
O_7, T_2	·0724	−·0237	+·0034	·0690
O_7, S_5	·0361	+·0131	+·0078	·0283
O_7, Ml_1	·0542	−·0010	−·0024	·0566
O_7, Mx_1	·1592	−·0283	+·0089	·1503
T_2, S_5	·3656	−·0145	+·0008	·3648
T_2, Ml_1	·1052	−·0187	−·0086	·1138
T_2, Mx_1	·1109	−·0308	−·0197	·1306
S_5, Ml_1	·1665	−·0215	−·0139	·1804
S_5, Mx_1	·2150	·0000	−·0123	·2273
Ml_1, Mx_1	·1776	−·0001	−·0132	·1908
Mean Numerical Difference		·0215	·0094	—

within the limits of random sampling, the value of the cross-correlations in terms of direct and homologous correlations in the case of absolute lengths.

General Conclusions.

In considering the structure of the skull, we should *a priori* expect that the dimensions and shapes of the individual bones of which it is constructed would be controlled by certain factors, capable of being statistically investigated when appropriate measurements have been made on the individual bones.

In the first place the skull is approximately symmetrical, having mirror symmetry about the "median sagittal plane." This approximate symmetry would appear to involve a correspondence in size and shape of homologous bones. We term the source of this the *factor of homology*, and measure it by the correlations between corresponding lengths and indices taken on the homologous bones.

Secondly, we note not only the approximate symmetry of a skull, but that one skull is larger than a second. We should anticipate that the size of one bone would be highly related with the size of a second. This leads us to a *factor of size*, which may be measured by the correlation of lengths taken on different bones on the same side, or on heterologous bones on opposite sides.

Thirdly, we should consider that the bones of the skull fit one another or that there must be a relationship of shapes. This leads us to a *factor of shape* measured by the correlation between the indices of bones.

Fourthly, we might anticipate that the shape of a cranial bone would be more influenced by an adjacent bone than by one not in contact with it. This leads us to a *factor of contiguity*. This is measured by the correlations between adjacent and non-adjacent bones.

(i) It has been long a desire of statisticians to obtain statistical constants independent of size. For this purpose the coefficient of variation was introduced by one of the present writers as an attempt to get over the difficulty that a large length has usually a larger absolute variation. It is shown on the present data that the correlation between absolute size and absolute variation is on the whole not as large as might be anticipated, i.e. about $+\cdot28$*, while the correlation of absolute length and coefficient of variation is $-\cdot12$. In other words, we reduce the relation of variation to size by more than a half by using the coefficient of variation, but it *over* corrects. The coefficient of variation varies rather inversely than directly as the size of the organ.

(ii) The fact that coefficients of variation are not equal even for two lengths measured on the same bone, led us to investigate whether variation was associated with the orientation of a measured length with regard to the fundamental planes of the skull. We divided up our data into broad classes, sagittal, transverse and vertical, with subclasses for lengths going in intermediate directions, and found that for coefficients of variation of absolute lengths there was least variation in the sagittal, most variation in the vertical, and an intermediate variation in the transverse direction. The mixed classes fitted well into this scheme. The same method was applied to the absolute variations (standard deviations), but the results were far less clear cut.

It will be seen from this result that the laying down of a given bone of an individual skull does not proceed equally in all directions—bone growth is not *isotropic*. This may be due to the existence of more than one centre of ossification or to other causes at present not studied.

(iii) Putting aside the like lengths on homologous bones, of the correlations of absolute lengths only 16 correlations out of 112 exceed $\cdot35$, the remaining 96 lie below $\cdot25$. Those exceeding $\cdot35$ are F_1 with F_2 and P_2, F_2 with P_2 and T_2 with S_5. It is clear that the factor of size—except for these bones, frontal, parietal, temporal and sphenoid—is not an all-important factor in regulating the relationship

* Cf. *Biometrika*, Vol. XXII. p. 339, footnote †.

between individual bones of the skull. The mean numerical correlation of the whole 112 pairs is only ·1751*. If this be compared with the mean correlation of the like absolute lengths on homologous bones, i.e. ·8198†, we see that the latter is 4·7 times the former, or the factor of homology is almost five times as intense as the factor of size.

(iv) While lateral influence is found in absolute lengths (see p. 424, (ii)), it has no appreciable influence on either the magnitude or variability of the indices of either heterologous or homologous bones (see pp. 426, (iii); 435—436).

(v) In the case of like characters of homologous bones, the factor of homology shows greater intensity on the facial bones, whether judged by indices or absolute lengths. The malar, maxillary and frontal bones have the highest correlation coefficients, or we may say that the face is more symmetrical than the brain-box. The sphenoid is exceptional.

The mean index correlation for homologous bones is ·7414, while that for absolute measurements of homologous bones is ·8198. Thus the factor of homology is somewhat more powerful in the case of absolute lengths—in directing size—than in that of indices, i.e. governing shape. But in both cases the factor of homology is far more intense than that of any other of the factors we have considered.

(vi) Confining ourselves to the relationship of indices of bones both on the same side, we endeavoured to measure the power of proximity in determining the correlation of indices. We divided our bones into non-contiguous, slightly contiguous, and contiguous groups. The numerical mean correlations were ·0397, ·0434 and ·0655 respectively, showing increasing mean correlation with a greater contiguity. If we pay regard to sign, we have for the three groups: ·0197, ·0145 and −·0164. The former result indicates that we may pool the slightly contiguous with the non-contiguous, giving non-contiguous and contiguous groups as ·0407 against ·0655. The latter result is more or less meaningless, owing to the idle character of sign in index correlations. We have shown in the course of the text that the whole system of contiguous bone indices might be reasonably looked upon as a sample from a population of correlation ·0655. Thus the maximum influence of the factor of adjacency may be represented by ·0655, as against the factor of homology ·7414, which is more than eleven times as great.

(vii) As far as shape—not size—is concerned, the magnitude of the homology factor is indicated by the fact that the cross-index correlations give numerically a mean of ·0656 for contiguous bones, while the direct index correlations for the same bones give ·0655, while for the non-contiguous bones the values are ·0456

* The L, L correlations are somewhat greater than the R, R correlations, probably significantly, but the difference is not of practical importance. The cross-correlations R, L and L, R are sensibly equal but somewhat less than the direct correlations. The differences are again so small that we have a further illustration of the power of the factor of homology: see foot of Table XXI, p. 458.

† Cf. *Biometrika*, Vol. XXII. p. 330, Table I.

and ·0397 respectively. It is thus on the average indifferent whether we correlate the index of a bone *A* with that of a bone *B*, contiguous or not with it on the same side, or with the index of a bone *B'* homologous to *B*.

Finally we can sum up as follows:

Average intensity of factor of homology:

For size ·8198, for shape ·7414.

Average intensity of factor of size for non-homologous bones:

For size ·1751*.

Average intensity of factor of contiguity at a maximum:

For size† ——, for shape ·0655.

From these numbers it will be clear to the reader that the only factor controlling the shape and size of the skull as a whole worthy of real consideration is that of homology. Homologous bones must grow both in shape and size alike—we have no suggestion to make of why they do so, we only note the fact that they do so. On the other hand, the factor for size is of small intensity; we cannot predict, except in a very few cases, the absolute size of one bone from another. Nor can we predict from the shape of an individual bone what will be the shape of a second, whether it be adjacent to it or not. It is as if bones spreading from their centres of ossification grew wholly individually, except that homologous bones grow alike. There is no fitting of adjacent bones to a standard pattern, no general control factor or factors (beyond that of homology) by which the skull as a whole is organised. There is no system of hereditary determinants guiding the shape and size of the cranial bones to predetermined forms. The ossification of the brain-box is merely a process of covering the brain with a bony surface, much as a plasterer covers a wall by dabs of plaster each independent of the size and shape of other dabs. Even the factor of homology may flow from the symmetry of the brain itself. It would not, we hold, be far from the truth to say that it is the brain which is inherited, the inheritance of size and shape of the skull itself is only an indirect cause of this more fundamental inheritance.

If the statements made above be even approximately true, what bearing have they on the craniometrician's processes? He has been seeking for some time for characters of the skull which are

(*a*) practically uncorrelated—independent of one another;

(*b*) racially differentiated.

The usual ethnographic characters of the anthropologists do not satisfy (*a*), however much they may be of value from the standpoint of (*b*). Not even in the case

* Even this reduces to ·1208, if we exclude the four exceptional cases of F_1 with F_2, F_1, F_2 with P_2 and T_2 with S_2.

† Not yet investigated.

of the indices usually provided by anthropologists is there independence*. From the data provided in the present paper, it is possible to select a system of lengths and indices with negligible intercorrelations. Will such lengths and indices satisfy (*b*) as well as (*a*)?

In other words, while in the individual skull there is no correlation of the growth of bones, will the general method of growth of the skull in different races be peculiar to each race? Have morphometric characters racial value? This can only be ascertained when they have been measured for additional races. But should they prove to have—as we anticipate—racial value, then the position of the Coefficient of Racial Likeness as a means of discriminating between races will be markedly strengthened. It needs a system of characters actually or practically independent.

We wish to thank most heartily Dr G. M. Morant for continuous help during the preparation of this paper.

* We may cite the following data from the present paper and that of Pearson and Davin (*Biometrika*, Vol. xvi. (1923), p. 348, Tabie V):

Mean Correlation of Indices	Morphometrical Indices	Ethnographical Indices
r Number	♂: $+\cdot0511 \pm \cdot0025$ 119	♂: $+\cdot1468 \pm \cdot0186$, ♀: $+\cdot1315 \pm \cdot0210$ 19 19

Thus in the case of the males the mean morphometric index correlation is only about one-third of the mean ethnographic (or anthropometric) index. The morphometric correlations (excluding homologous indices) include 61 cases of unilateral and 58 of cross correlations.

MISCELLANEA.

(i) The Distribution of the Difference between the Extreme Observation and the Sample Mean in Samples of n from a Normal Universe.

By A. T. McKAY, D.Sc.

The purpose of the present paper is to discuss certain points relating to statistics formed by arranging the observations of a random sample from a normal universe in order of magnitude and constructing linear functions thereof. In particular, a method of determining the significance of the difference between the highest observation and the sample mean is developed.

I. Let $x_1, x_2, \ldots x_n$, arranged in order of ascending magnitude, be a random sample of n from a normal universe with mean m and unit standard deviation. Consider the statistic X formed by taking a linear function of the first r observations, i.e.

$$X = \sum_1^r a_r x_r, \qquad a_r \neq 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Suppose α and β are the characteristic function parameters of the mean $\bar{x}$ of the n observations and of the function X respectively; ϕ is the complete distribution function of X in samples of n and $\sigma(n)$ is its standard deviation; $\exists$ denotes the term "characteristic function of..."; and $\epsilon = \sum_1^r a_k$. Then if S denotes the distribution surface of $\bar{x}$ and X,

$$\exists(S) = e^{\alpha^2/2n + \alpha\beta\epsilon/n + m\alpha + m\epsilon\beta} \exists_\beta (\phi_{m=0}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

To prove this theorem consider the case in which $n=4$ and $r=2$. For this we have

$$\exists(S) = C \int_{-\infty}^{\infty} e^{\alpha z/n - \frac{1}{2}(z-m)^2}\, dz \int_{-\infty}^{z} e^{\alpha y/n - \frac{1}{2}(y-m)^2}\, dy \int_{-\infty}^{y} e^{(\alpha/n + a_2\beta)x - \frac{1}{2}(x-m)^2}\, dx$$
$$\int_{-\infty}^{x} e^{(\alpha/n + a_1\beta)u - \frac{1}{2}(u-m)^2}\, du \quad \ldots\ldots (3).$$

Changing the variable in the last integral on the right-hand side of (3) by writing $u - m - \frac{\alpha}{n} = t$ we find

$$\exists(S) = C e^{\alpha^2/2n^2 + \alpha m/n + a_1\alpha\beta/n} \int_{-\infty}^{\infty} (\ldots) \int_{-\infty}^{x-m-\alpha/n} e^{-t^2/2 + a_1\beta t}\, dt \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Proceeding similarly by changing the variables of integration in turn we derive, in reverting to our original symbols,

$$\exists(S) = e^{\alpha^2/2n + \alpha\beta t/n + \alpha m + m\beta\epsilon} \int_{-\infty}^{\infty} e^{-z^2/2}\, dz \int_{-\infty}^{z} e^{-y^2/2}\, dy \int_{-\infty}^{y} e^{-x^2/2 + a_2\beta x}\, dx \int_{-\infty}^{x} e^{-u^2/2 + a_1\beta u}\, du$$
$$\ldots\ldots (5).$$

This latter expression is in agreement with the special case of (2) and the general theorem follows by induction.

Let us now seek the analytical expression for the surface S. Equation (2) may of course be written

$$\exists(S) = e^{\alpha^2/2n + m\alpha + \alpha\beta\epsilon/n} \exists_\beta \phi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6),$$

so that interpreting first with respect to α and writing the corresponding variable x and the other variable y we have

$$\mathfrak{Z}_\beta S(x,y) = \sqrt{\frac{n}{2\pi}}\, e^{-\frac{1}{2}n(x-m-\beta\epsilon/n)^2}\, \mathfrak{Z}_\beta(\phi) \quad \ldots\ldots(7)$$

$$= \sqrt{\frac{n}{2\pi}}\, e^{-\frac{1}{2}n(x-m)^2 - \beta^2\epsilon^2/2n}\, \mathfrak{Z}_\beta \phi\,(y-\epsilon x+\epsilon m) \quad \ldots\ldots(8),$$

whence

$$S(x,y) = \sqrt{\frac{n}{2\pi}}\, e^{-\frac{1}{2}(x-m)^2} \times e^{-\frac{1}{2n}\epsilon^2 \frac{d^2}{dy^2}} \phi\,\{y-\epsilon\,(x-m)\} \quad \ldots\ldots(9).$$

The correlation coefficient R is derived from (6) by taking logarithms, differentiating with respect to α, β and $\alpha\beta$, and writing $\alpha=\beta=0$. Whence

$$R = \frac{\epsilon}{\sqrt{n}\,\sigma\,(n)} \quad \ldots\ldots(10).$$

Further, if $E(y)$ denotes the expected value of y, then from (6) the regression curve of y on x is

$$\bar{y} = \frac{\left\{\frac{\alpha\epsilon}{n} + E(y)\right\}}{e^{\alpha^2/2n + m\alpha}}\, e^{\alpha^2/2n + m\alpha} \quad \ldots\ldots(11),$$

provided the arrays in the numerator and denominator are interpreted independently. Hence

$$\bar{y} = \frac{\left\{E(y) - \frac{\epsilon}{n}\frac{d}{dx}\right\}}{e^{-n/2(x-m)^2}}\, e^{-\frac{1}{2}n(x-m)^2} \quad \ldots\ldots(12),$$

$$\bar{y} = E(y) + \epsilon\,(x-m) \quad \ldots\ldots(13).$$

Similarly the regression curve of x on y is derived from

$$\bar{x} = \frac{\left(m + \frac{\beta\epsilon}{n}\right) \mathfrak{Z}_\beta(\phi)}{\mathfrak{Z}_\beta(\phi)} \quad \ldots\ldots(14),$$

$$\bar{x} = \left(m - \frac{\epsilon}{n}\frac{d}{dy}\log\phi\right) \quad \ldots\ldots(15).$$

II. Let us now consider the linear statistical function formed by subtracting the sample mean from the function X, i.e.

$$u = (X - \bar{x}) \quad \ldots\ldots(16).$$

The characteristic function for the distribution of u is to be found from (6) by writing $-\beta$ for α and then regarding β as referring to u. Thus

$$\mathfrak{Z}_\beta f(u) = e^{\frac{1}{2n}(1-2\epsilon)\beta^2 - m\beta}\, \mathfrak{Z}_\beta(\phi) \quad \ldots\ldots(17),$$

whence

$$f(u) = e^{\frac{1}{2n}(1-2\epsilon)\frac{d^2}{du^2}} \phi(u+m) \quad \ldots\ldots(18).$$

We note also from (17) that if K_p denotes the pth semi-invariant of the distribution of u and K_p' refers similarly to the distribution of X, then

$$\left.\begin{aligned} &\text{(Mean)} && K_1 = K_1' - m \\ &\text{(Variance)} && K_2 = K_2' + \frac{(1-2\epsilon)}{n} \\ & && K_p = K_p', \quad p = 2, 3, \ldots \end{aligned}\right\} \quad \ldots\ldots(19).$$

III. We shall now proceed to apply some of the foregoing results to the case when X = highest observation of the sample of n. Thus the ϵ associated with equation (1) is unity. Since no generality will be lost by so doing, we shall take $m=0$ in what follows. The means and standard

deviations of the distribution of X for various sizes of samples, which have already been given elsewhere*, enable us to construct the following table showing the standard deviation of $u=(X-\bar{x})$ and the correlation coefficient R for X and $\bar{x}$.

n	S.D. of u	R
2	·6179	·856
5	·4976	·668
10	·4943	·539
20	·4751	·426
60	·4358	·284
100	·4176	·233
200	·3946	·176
500	·3677	·121
1000	·3500	·090

It is noteworthy that even for samples as large as 60 R is still quite moderate.

Let us now discuss the distribution of u more closely. It is obvious firstly that

$$\phi(x)=\frac{n}{\sqrt{2\pi}}e^{-x^2/2}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-x^2/2}dx\right)^{n-1} \quad \text{.....(20).}$$

So from (18), if $F_n(x)$ is the probability of a sample of n occurring in which u is less than x,

$$F_n(x)=e^{-\frac{1}{2n}\frac{d^2}{dx^2}}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-x^2/2}\,dx\right)^{n} \quad \text{.....(21),}$$

and

$$f(x)=F_n'(x)=\frac{n}{\sqrt{2\pi}}e^{-\frac{1}{2n}\frac{d^2}{dx^2}}e^{-x^2/2}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{1}{2}x^2}\,dx\right)^{n-1} \quad \text{.....(22).}$$

Now suppose we proceed as though to expand the right-hand side of (22) by Leibnitz's Theorem, i.e.

$$F_n'(x)=\frac{n}{\sqrt{2\pi}}e^{-\frac{1}{2n}(D_1+D_2)^2}e^{-x_1^2/2}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x_2}e^{-x^2/2}\,dx\right)^{n-1} \quad \text{.....(23),}$$

where $D_1=\dfrac{\partial}{\partial x_1}$ and $D_2=\dfrac{\partial}{\partial x_2}$. Whence

$$F_n'(x)=\frac{n}{\sqrt{2\pi}}e^{-\frac{1}{2n}(D_1^2+D_2^2)}\times e^{-\frac{1}{2}(x_1-D_2/n)^2}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x_2}e^{-x^2/2}\,dx\right)^{n-1} \quad \text{.....(24).}$$

Now

$$e^{-\frac{1}{2n}D_1^2}\times e^{-\frac{1}{2}(x_1-D_2/n)^2}=\sqrt{\frac{n}{n-1}}\,e^{-\frac{1}{2}\frac{n}{n-1}(x_1-D_2/n)^2} \quad \text{.....(25).}$$

Substituting (25) in (24) and making one or two obvious steps we find

$$F_n'(x)=\frac{n}{\sqrt{2\pi}}\sqrt{\frac{n}{n-1}}\,e^{-\frac{1}{2}\frac{n}{n-1}x_1^2}\times e^{-\frac{1}{2}(n-1)D_2^2}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x_2+x_1/(n-1)}e^{-x^2/2}\,dx\right)^{n-1} \quad \text{...(26).}$$

Now we note

$$\left\{D_2^k\left(x_2+\frac{x_1}{n-1}\right)^m\right\}_{x_2=x_1=x}$$
$$=\left(\frac{n}{n-1}\right)^{m-k}D^k x^m$$
$$=\left\{\left(\frac{n-1}{n}\right)D\right\}^k\left(\frac{nx}{n-1}\right)^m.$$

* *Tables for Statisticians and Biometricians*, Part II. p. cxiv.

Thus, if ψ and χ are any two integral functions regular at the origin,

$$\left[\psi(D_2)\,\chi\left(x_2+\frac{x_1}{n-1}\right)\right]_{x_2=x_1=x}=\psi\left(\frac{n-1}{n}D\right)\chi\left(\frac{nx}{n-1}\right) \quad \text{.....(27).}$$

Applying the latter theorem to (26),

$$F_n'(x)=\frac{n}{\sqrt{2\pi}}\sqrt{\frac{n}{n-1}}\,e^{-\frac{1}{2}\frac{n}{n-1}x^2}\times e^{-\frac{1}{2}\frac{n-1}{n^2}D^2}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x\frac{n}{n-1}}e^{-x^2/2}\,dx\right)^{n-1} \quad \text{.....(28),}$$

$$F_n'(x)=\frac{n}{\sqrt{2\pi}}\,e^{-\frac{1}{2}\frac{n}{n-1}x^2}\sqrt{\frac{n}{n-1}}\,F_{n-1}\left(\frac{n}{n-1}x\right) \quad \text{.....(29).}$$

We now consider the case for $n=2$. To find the distribution of $u=\frac{x_2-x_1}{2}$ (where x_2 is the larger) we must integrate

$$\frac{1}{2\pi}e^{-\frac{1}{2}(x_1^2+x_2^2)}\,dx_1\,dx_2 \quad \text{.....(30)}$$

over the field for which

$$\left.\begin{aligned} u\leqslant\frac{x_2-x_1}{2}<u+\delta u\\ 0<(x_2-x_1)\end{aligned}\right\} \quad \text{.....(30}^{bis}\text{).}$$

Write $(x_2-x_1)=2w$ and $(x_2+x_1)=2v$. Whence (30) becomes

$$\frac{4e^{-(w^2+v^2)}}{\pi}\,dw\,dv \quad \text{.....(31),}$$

and (30^{bis})

$$\left.\begin{aligned} u\leqslant w\leqslant u+\delta u\\ 0<w\end{aligned}\right\} \quad (31^{bis}).$$

Whence the distribution of u is

$$f_2(u)\equiv F_2'(u)=\frac{2}{\sqrt{\pi}}\,e^{-u^2}, \qquad u>0 \quad \text{.....(32).}$$

This represents one-half a normal curve. We note that the characteristic function of this distribution is

$$\exists f_2(u)=\frac{e^{\beta^2/2}}{\sqrt{\pi}}\int_{-\infty}^{\beta}e^{-t^2/4}\,dt \quad \text{.....(32}^{bis}\text{).}$$

Further, using (29) and (32),

$$f_3(u)=\frac{3\sqrt{3}}{\pi}\,e^{-\frac{3}{4}u^2}\int_0^{\frac{3}{2}u}e^{-t^2}\,dt.$$

Other cases can be found similarly, but they are of no interest since the integrals cannot be evaluated simply.

Let us now return to equation (29). It is easy to see that when x is very small

$$F_n(x)=\frac{n}{2}\,n^{\frac{n}{2}}\,\frac{x^{n-1}}{(\sqrt{2\pi})^{n-1}}, \qquad n>2 \quad \text{.....(33).}$$

Again,
$$P_n(x)=\int_x^{\infty}F'(x)\,dx\sim\frac{n}{\sqrt{2\pi}}\int_{x\sqrt{n/(n-1)}}^{\infty}e^{-t^2/2}\,dt \quad \text{.....(34),}$$

provided $P_{n-1}\left(\frac{n}{n-1}x\right)$ is small compared with unity. But

$$P_{n-1}\left(\frac{n}{n-1}x\right)\leqslant\frac{(n-1)}{\sqrt{2\pi}}\int_{x\frac{n}{n-1}}^{\infty}\sqrt{\frac{n-1}{n-2}}\,e^{-t^2/2}\,dt \quad \text{.....(35).}$$

Whence $P_n(x)$ may be found approximately from equation (34) provided the expression on the

right-hand side of (35) is small compared with unity. The latter is always less than $P_n(x)$. So we may conclude that (34) can always be used for determining the low percentage points*.

In a given case the significance of departure of the highest observation from the mean of the sample is derived directly from equation (34).

Owing to the fact that $\frac{dP}{dx}$ is small when x is large it is clearly more accurate to determine $P_n(x)$ when x is given than to construct a table giving x for standard values of ·05, ·01, etc. of $P_n(x)$. We shall now consider one or two examples of the use of the statistical function u = highest observation minus the mean of the sample.

Example 1. K. Pearson† gives the capacities of 17 Moriori skulls, the highest value of which (1630) appears anomalous, and discusses the rejection of this observation on the basis of three different methods. The mean and standard deviation of the sample are respectively 1405·8 and 97·83. Thus $u=(1630-1405{\cdot}8)/\sigma$, where σ is the standard deviation of the parent universe. Let us assume that this latter is closely given by the sample estimate, and that the universe is normal. Then $u=(1630-1405{\cdot}8)/97{\cdot}83=2{\cdot}2917$. Also $u\sqrt{\tfrac{17}{16}}=2{\cdot}3622$. Whence from equation (34) $P_{17}(u)=17\times 0{\cdot}009084=0{\cdot}154$. We should therefore conclude that 1630 was not an anomalous value. Applied to the same data Irwin's test‡ gives a probability of 0·134. Using E. S. Pearson's table§ for the percentage limits of the range, a probability somewhat greater than 0·10 is obtained. The three tests show good agreement.

Example 2. In the course of routine testing of a standard leather product of a tannery five parallel tests yielded the following values for the hide substance content of the leather specimens:

32·44, 36·45, 39·64, 40·13, 41·09.

The first observation appears unduly low.

The question therefore facing the tannery chemist is whether it is worth while instituting inquiries as to why such an anomalous specimen has occurred. Long experience of the product in question has established a value of 2·226 for the standard deviation. Whence

$$u=(37{\cdot}95-32{\cdot}44)/2{\cdot}226=2{\cdot}475.$$

Also $\sqrt{\tfrac{5}{4}}\,u=2{\cdot}767$, whence $P_5(u)=5\times 0{\cdot}00283=0{\cdot}0142$.

We should conclude from this that the observation is anomalous and inquiries could justifiably be instituted. Irwin's test applied to the same data gives a value of 0·06 for the probability, whilst using the range as criterion E. S. Pearson's table gives a value of 0·05. The latter two tests are just about *on* the usually accepted boundaries of significance. Accordingly on the basis of these one would hesitate to regard the observation as anomalous. It is of course difficult to decide which test gives the surest guidance, though the author's experience of the material in question leads him to believe that the value of 32·44 is definitely anomalous.

[Note. Some readers may be interested in learning what information the $P_{\lambda_{21}}$ criterion provides with regard to the above *Example* 2. That criterion gives equal weight to every member

* The same kind of argument can always be applied in the case of the distribution of the highest observation of a sample from any parent universe. Let $f(x)$ be the parent universe, then the distribution of the highest observation in a sample of n is

$$F_n'(x)=nf(x)\left(\int_{-\infty}^{x} f(x)\,dx\right)^{n-1},$$

$$\therefore P_n(x)=n\int_{x}^{\infty} f(x)\,dx,$$

provided $\frac{(n-1)}{n}P_n(x)$ is small compared with unity. This will always be the case in the vicinity of the levels of significance which are usually employed.

† *Loc. cit.* Tables, Part II. p. cviii.

‡ *Loc. cit.* Tables, Part II. p. 160.

§ *Biometrika*, Vol. xxiv. p. 416 (1932).

of the sample; this is also true of the β_2 test but not of Irwin's or McKay's test. The P_{λ_3} criterion gives the value ·006 to the probability, if we use the "established value" 2·226 for the standard deviation. Thus we infer that the outlier ought to be rejected, or that the sample has not been drawn at random from a normal population. On the other hand, if we adopt the standard deviation 3·165 given by the sample itself, we find the probability to be ·42, or no question of throwing out the outlier arises. But inquiring whether the standard deviation 3·165 was a reasonable random sample from a population of standard deviation 2·226, we find that the probability of a sample with its standard deviation $\geqslant 3{\cdot}165$ is only ·06, which agrees with the results of Irwin and E. S. Pearson's tests. It would seem accordingly that given the value 3·165 the sample is quite probable, but that such a standard deviation verges on the improbable if the established value of the parental population is 2·226. ED.]

(ii) Heredity, mainly Human.

BY ELDON MOORE.

Chapman and Hall, Ltd. London, 1934. Price 15*s*.

The author of this volume writes for "the educated and intelligent reader who knows nothing of the subject, but wishes to understand its main outlines without intending to pursue it much further." The book covers a wide field, including chapters on genes and the mechanism of heredity, on the endocrine glands, on physique and character, on hybrids, on inbreeding, on blood groups and on the inheritance of amentia and of ability; some investigations on inheritance in twins are described and an interesting section is concerned with intelligence and social class; the latter part of the volume provides a general review of such subjects as evolution, natural selection, mutation, Lamarckism, etc.

It requires courage to set out on so ambitious a programme and the author is to be congratulated on having produced a volume which clearly portrays the intense interest, academic and practical, attaching to the science of Genetics to-day; he has succeeded in suggesting the general trend of progress in this field and the volume will certainly stimulate interest in the subject.

Writing admittedly for the reader "who knows nothing of the subject" carries with it a special responsibility with regard to the accuracy of the facts given; a number of statements are made which will arrest the critical mind of a reader who has some knowledge and may perhaps shake his confidence. To give one instance on pp. 69—70: "The endocrine constitution...plays a part...in three of the commonest human ills with a genetic basis—asthma, cancer and tuberculosis, which are in some way related to one another....Cancerous patients are very seldom tubercular* while the tuberculous are but rarely afflicted by cancer....Tuberculosis is markedly rare both in asthmatic patients and in their families." All these statements are most challenging and are far from being proven facts; the genetic basis of cancer in man has not been established, nor that its sufferers, as such, carry any insurance against tuberculosis. The genetic basis of asthma must surely be regarded as standing upon precarious ground until we know more of the incidence of asthma in the general population and the incidence of all the other allergical diseases which may or may not replace it.

On what evidence does the author state that children who will later develop Huntington's chorea very frequently have unusually small heads? Other similar criticisms might be given but we can recommend a volume which contains so much interesting suggestion, and only regret the occasional lack of caution in the too definite presentation of unproven facts.

* [As a matter of fact cancer of the lungs is often found to develop on the site of healed tubercular lesions. See Karl Pearson, "Report on certain Cancer Statistics," *Archives of the Middlesex Hospital*, pp. 8—12. Nor can we yet assert that the tendency to cancer is certainly hereditary. See *ibid.* pp. 5—8; but compare Brenda Stoessiger, "Inheritance of Duration of Life and Cause of Death," *Annals of Eugenics*, Vol. v, "Cancer," pp. 148—150, especially p. 150. See also Percy Stocks, "A Cooperative Study of 450 Cancer Patients and an equal Number of Controls," *ibid.* Vol. v. pp. 274—278 and 280. ED.]

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